Improved Multi-Particle Model for Dynamics Analysis of Arbitrary-Shaped Thin Flexible Structure

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Abstract

The conventional multi-particle model, which has been developed for fast analysis of dynamic behaviour of the right rectangular-shaped thin flexible structure, is improved so as to be applicable for arbitrary-shaped mesh. This extension is done by introducing a few additional elastic parameters, and the resulting extended formulation enables any complicated shape and arbitrary mesh membrane to be dealt within a scope of multi-particle analysis. This method is expected to give us a reasonably good time-cost and accuracy, in solving the thin flexible structure dynamics of ISAS’s solar-sail mission.

任意形状に対応した多粒子系モデルによる薄膜挙動解析手法について
津田 雄一 (ISAS/JAXA)

摘要

薄膜構造物の過渡的な挙動を高速で解析する手法として、薄膜をバネマスで置き換える多粒子系モデルによる手法が提案されているが、これは合理的なバネマスの配置とそのパラメータの決定の仕方に制限があり、複雑な形状・折り目を有する薄膜の挙動解析は困難であった。本稿では、弾性パラメータの数を若干増やすことにより、任意形状・任意メッシュの薄膜を多粒子系モデルで表現する手法を提案する。本手法は、ソーラーセールミッション等、膜の挙動と姿勢制御がカップルする問題において、リーズナブルな計算速度と精度を期待できるものである。

1 Introduction

The conventional multi-particle model (MPM), which was originally developed in ISAS for the fast analysis of transitional dynamics of thin flexible structure[1], is extended so as to be applicable for practical problems arisen in the study of ISAS’s solar sail deep space explorer mission. The MPM substitutes the elements of flexible structure for particles connected by springs and dashpots, to drastically reduce the calculation time-cost while keeping a certain level of accuracy.

The original MPM divides a thin structure into right-rectangular mesh, and gives us a unique set of consistent spring and mass constants. However, because of the relatively too tight restriction as to the formulation (such as the shape of mesh, Poisson’s ratio, etc.), it is not always applicable to the practical problems arisen in the study of thin flexible structures[2].

This paper extends the conventional multi-particle model (MPM), and re-formulate so that it can deal with an arbitrary shape, arbitrary mechanical parameters membrane without violating any consistency conditions of structural mechanics. This extensions enable the transitional dynamics analysis of arbitrary-shaped thin flexible structure with a reasonably good time-cost and accuracy, so that we can use it for the study of the solar-sail mission.

This paper shows the formulation of the improved MPM, and examines its validity by a few simulations.

2 Conventional Multi-Particle Model

The mesh for the conventional MPM is shown in Fig.1. The consistent spring parameters are determined by applying the principle of virtual work on □ABCD and ◇CDFE, but with this formulation, we have the following limitations and shortcomings:

- The shape of mesh must be right-rectangular divided by its diagonal line. The other shape violates the consistency conditions.
- The Poisson’s ratio(ν) must be 1/3 to satisfy the consistency conditions. Additionally, even when a Poisson’s ratio of 1/3 is used, the numerical result doesn’t realize the deformation
corresponding to $\nu = 1/3$.

- The consistency along directions other than $x$ and $y$ axes are not taken into account.

These limitations are due to the lack of elastic parameters that should have been taken into account. The situation becomes worse when a deformed rectangular is chosen as a mesh. In such a case, the consistent set of elastic parameters is not obtained because it requires more consistency conditions than the number of parameters.

In the following sections, the conventional MPM will be improved so that it can be applied to any shapes of thin structure, while holding all the structural mechanics conditions.

### 3 Improved Multi-Particle Model

#### 3.1 Elastic Matrix Derivation

**Basic Idea** To make MPM work with any shapes of element, the elastic parameters for each triangle element ($3 \times 3$ dimension) is extended from diagonal matrix eq.(1) to generic non-diagonal matrix eq.(2).

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
\end{bmatrix} =
\begin{bmatrix}
k_1 & k_2 & k_3 \\
k_1 & k_2 & k_3 \\
k_{11} & k_{12} & k_{13} \\
k_{21} & k_{22} & k_{23} \\
k_{31} & k_{32} & k_{33} \\
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\end{bmatrix}
\]

(1)

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
\end{bmatrix} =
\begin{bmatrix}
k_{11} & k_{12} & k_{13} \\
k_{21} & k_{22} & k_{23} \\
k_{31} & k_{32} & k_{33} \\
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\end{bmatrix}
\]

(2)

In this case, each parameter no longer represents the spring constant between two mass points, but 9 parameters in one elastic matrix as a whole represents the elasticity characteristics of a triangle element (Fig.2.)

**Displacement Function** The conventional MPM uses strain energy relations for its derivation, but here we use a displacement function approach to obtain the parameters $k_{11}, \ldots, k_{33}$ of (2).

Fig.3 shows an element for improved MPM. The triangle $\triangle OAB$ can be arbitrary shape. Because we now consider the displacement of the element, only 3 parameters ($u_A$, $u_B$ and $v_B$) are taken into account. When the displacement inside an element is defined by the following displacement functions;

\[
\begin{align*}
u(x, y) &= a_1 x + a_2 y \\
v(x, y) &= b_1 x + b_2 y
\end{align*}
\]

$a_1, a_2, b_1, b_2$ can be obtained from the relations shown in Fig.3 and are expressed in the matrix form as follows;

\[
f(x, y) = \begin{bmatrix} u \\ v \end{bmatrix}
\]

\[
def=N(x,y)u
\]

(3)

**Total Distortion** Using (4), the total distortion for the triangle element is calculated as follows;

\[
\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{1}{x_B} & 0 & 0 \\ 0 & \frac{1}{y_B} & 0 \\ -\frac{xy}{x_B y_B} & 0 & 0 \end{bmatrix}
\begin{bmatrix} u_A \\ u_B \\ v_B \end{bmatrix}
\]

(5)
Stress-Strain Relation  The stress-strain relation is:

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = D \epsilon = D \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$  \hspace{1cm} (6)

From the structural mechanics theory, if we assume the plane-stress and isotropic material conditions, then $D$ becomes:

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$  \hspace{1cm} (7)

Elastic Matrix  Now we apply the principle of virtual work by equalizing an internal force with an external force;

$$(du)^T F_{OAB} = \int d\varepsilon^T \sigma dV$$  \hspace{1cm} (8)

where $u = (u_A \ u_B \ v_B)^T$. $F_{OAB} = (f_{Ax} \ f_{Bx} \ f_{By})^T$ is a vertex force corresponding to the displacement $u$. From (5)-(8), $F_{OAB}$ can be calculated as follows;

$$F_{OAB} = \left( \int B^T D B dV \right) u$$

$$= K_{OAB} u$$  \hspace{1cm} (9)

If an uniform thickness membrane is assumed, the concrete expression of $K_{OAB}$ can be approximately obtained as follows;

$$K_{OAB} = \int B^T D B dV \approx B^T D t S$$

$$= \frac{E}{2(1-\nu^2)} \begin{bmatrix} \frac{\nu t B}{\nu A} + \frac{1-\nu}{2} \frac{\nu^2 t B}{\nu A} B & -\frac{1-\nu}{2} \frac{\nu^2 t B}{\nu A} B & \nu \\ -\frac{1-\nu}{2} \frac{\nu^2 t B}{\nu A} B & \frac{1-\nu}{2} \frac{\nu^2 t B}{\nu A} B & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$  \hspace{1cm} (10)

where $t$ is the thickness, $S$ the area of the triangle element.

To fit to the MPM formulation, we must relate the force at each vertex to the expansion of the branches of an element. Let $Q$ denote the coordinates transformation matrix from the orthogonal coordinates to the branch directions;

$$Q = \begin{bmatrix} -\frac{x_B-x_A}{l_1} & \frac{x_B-x_A}{l_2} & \frac{x_B-x_A}{l_3} \\ 0 & \frac{y_B-y_A}{l_2} & \frac{y_B-y_A}{l_3} \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$  \hspace{1cm} (11)

where $l_1$, $l_2$, $l_3$ are the length of the branches AB, OA, OB in Fig.3. Then the relation between the force along branches $F = (f_1 \ f_2 \ f_3)^T$ and the expansion of the branches $\delta = (\delta_1 \ \delta_2 \ \delta_3)^T$ (See Fig.2 for illustrative definition) can be written as follows;

$$F = K \delta$$  \hspace{1cm} (12)

$$K = Q^T K_{OAB} Q$$  \hspace{1cm} (13)

From (10),(11) and (13), we can calculate the elements of $K$.

3.2 Damping Coefficient Derivation  A damping coefficient is also an essential parameter to obtain a stable and rational solution for MPM. If we define $p$ as a force per unit volume, the virtual work is calculated from (4) as;

$$-df^T p = -(du)^T \left( N^T p \right)$$  \hspace{1cm} (14)

so that the principle of virtual work yields;

$$F_{OAB} = - \int N^T p dV$$  \hspace{1cm} (15)

The physically rational damping force must be proportional to the in-plane displacement velocity (Fig.4);

$$p = -\frac{\partial}{\partial t} f = -\mu \frac{\partial}{\partial t} N u$$  \hspace{1cm} (16)

From (15) and (16), the vertex force in the orthogonal frame becomes;

$$F_{OAB} = \mu \left( \int N^T N dV \right) u \overset{\text{def}}{=} C_{OAB} \hat{u}$$  \hspace{1cm} (17)

where $C_{OAB}$ can be calculated concretely as follows;

$$C_{OAB} = \mu \left( \int N^T N dV \right) = \frac{\mu S}{12} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$  \hspace{1cm} (18)

Finally the vertex force along the branch directions is obtained as follows;

$$F = Q^T C_{OAB} Q \overset{\text{def}}{=} C \hat{\delta}$$  \hspace{1cm} (19)

From (11),(18) and (19), we can easily calculate the elements of $C$.  

![Figure 4: Variables for Damping Derivation](image)
3.3 Mass Matrix Derivation

To consider the inertial force acting on an triangle element, one must re-define the displacement function to include the 3 dimensional directions, 3 vertices so as to take into account the in-plane and outer-plane translational motion as well as the deformation of the element (Fig. 5). The re-defined displacement functions are:

\[ u(x, y) = a_1x + a_2y \]
\[ v(x, y) = b_1x + b_2y \]
\[ w(x, y) = c_1x + c_2y \]

where \( w(x, y) \) is a displacement along z axis (outer-plane). Because the displacement of O,A,B is \( f = (u_i \ v_i \ w_i)^T \). To consider the inertial force acted on an the membrane, one must re-define the displacement function to include the 3 dimensional directions, 3 vertices so as to take into account the in-plane and outer-plane translational motion as well as the deformation of the element (Fig. 5). The re-defined displacement functions are:

\[
\mathbf{f}(x, y) = \begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix} = \begin{bmatrix}
    N_i \quad N_j \quad N_m
\end{bmatrix} \begin{bmatrix}
    u_i \\
    v_i \\
    w_i
\end{bmatrix}
\]

where \( I \) is a 3 \times 3 unit matrix, \( N_i = (a_i + \beta_i + \gamma_i)/(2S) \), \( \alpha_i = x_i y_m - x_m y_i \), \( \beta = y_j - y_m \), \( \gamma_i = x_m - x_j \).

Then (20) can be changed to the following formula:

\[
\mathbf{f}(x, y)^T = \begin{bmatrix}
    N_i \\
    N_j \\
    N_m
\end{bmatrix} \begin{bmatrix}
    u_i & v_i & w_i \\
    u_j & v_j & w_j \\
    u_m & v_m & w_m
\end{bmatrix}
\]

\[ \mathbf{f}(x, y)^T = \mathbf{N} \mathbf{X} \] (21)

Next, consider the inertial force acted on an unit volume:

\[
p = -\rho \frac{\partial^2}{\partial t^2} \mathbf{f}(x, y) = -\rho \frac{\partial^2}{\partial t^2} \mathbf{N} \mathbf{X}
\] (22)

where \( \rho \) is a density of the element. From (15), the vertex force along orthogonal axes can be obtained as follows:

\[
\mathbf{F}_{OAB} = \rho \left( \int \mathbf{N}^T \mathbf{N} dV \right) \ddot{\mathbf{X}} \quad \text{def} \quad \ddot{\mathbf{X}}
\] (23)

After some manipulations using (21) and (23), we can obtain the following simple expression for \( \mathbf{M} \):

\[
\mathbf{M} = \frac{\rho t S}{3} \begin{bmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1
\end{bmatrix}
\] (24)

This result indicates that the consistent mass matrix must not be a diagonal matrix, as is used in the conventional MPM.

3.4 Equation of Motion

To give a good perspective for constructing the equation of motion for the full model of a membrane, let us consider first the equation of motion for only one element. Let \( \mathbf{X} \) denote the positions matrix which includes 3 dimensional positions of all the vertices of an element:

\[
\mathbf{X} = \begin{bmatrix}
    x_O & y_O & z_O \\
    x_A & y_A & z_A \\
    x_B & y_B & z_B
\end{bmatrix}
\] (25)

and let \( \mathbf{Y} \) denote the branches direction matrix:

\[
\mathbf{Y} = \begin{bmatrix}
    x_B - x_A & y_B - y_A & z_B - z_A \\
    x_A - x_O & y_A - y_O & z_A - z_O \\
    x_B - x_O & y_B - y_O & z_B - z_O
\end{bmatrix}
\] (26)

\[
(20) \mathbf{X} \text{ and } \mathbf{Y} \text{ are related by a branch-node conversion matrix } \mathbf{R} \text{ as:}
\]

\[
\mathbf{Y} = \begin{bmatrix}
    0 & -1 & 1 \\
    -1 & 1 & 0 \\
    -1 & 0 & 1
\end{bmatrix} \mathbf{X} \text{ def } = \mathbf{R} \mathbf{X}
\] (27)

Note that the inverse relation \( \mathbf{X} = \mathbf{R}^T \mathbf{Y} \) always holds.

If \( \Delta \mathbf{Y} \) denote the displacements matrix (each row vector represents the displacement direction and magnitude of a branch), then the equation of motion for one element can be written as follows:

\[
\mathbf{M} \ddot{\mathbf{X}} = -\mathbf{R}^T \mathbf{K} \Delta \mathbf{Y} - \mathbf{R}^T \mathbf{C} \dot{\mathbf{Y}}
\] (28)

The equation of motion for a full membrane is obtained by superposing each element’s equation of motion, just summing up each vertex information one by one, and takes the following form:

\[
[M][\ddot{\mathbf{X}}] = -[R]^T [K][\Delta \mathbf{Y}] - [R]^T [C][\dot{\mathbf{Y}}]
\] (29)

where a bracket [ ] represents the superposed variables over a whole membrane.
Table 1: Comparison of Conventional and Improved MPM

<table>
<thead>
<tr>
<th></th>
<th>Conventional MPM</th>
<th>Improved MPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass Matrix</td>
<td>Diagonal</td>
<td>Symmetric</td>
</tr>
<tr>
<td>Elastic Matrix</td>
<td>Diagonal</td>
<td>Symmetric</td>
</tr>
<tr>
<td>Damping Matrix</td>
<td>? (Not defined)</td>
<td>Symetric, Proportional to in-plane deformation rate</td>
</tr>
<tr>
<td>Non-0 Elements of Each Matrix</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>Element’s Shape</td>
<td>Isosceles right-angled triangle</td>
<td>Arbitrary triangle</td>
</tr>
</tbody>
</table>

4 Performance of Improved MPM

The comparison of the conventional and improved MPM is shown in Tab.1. The improved MPM requires 3 times more non-zero elements for each material parameter matrix, which means the calculation cost gets only 3 times worse than the conventional formulation. Instead, the improved MPM allows an arbitrary shaped, arbitrary material parameters element without so much changing the original MPM formulation.

To validate the improved MPM works properly, a Poisson’s effect is simulated both for the conventional and improved MPM. The model tested here is a simple right rectangular consisting of four elements(Fig.6). The shape of the elements is selected so as to be applicable for the conventional MPM formulation as well as improved one.

The Poisson’s ratio used for this test is set as \( \nu = 1/3 \), which is an only consistent value for the conventional MPM. The result is shown in Fig.7. Although the Poisson’s ratio of 1/3 is selected, the result of the conventional MPM shows the Poisson’s ratio of almost 1.0, which is one of the problems of the conventional MPM as is mentioned at first in this paper. On the other hand, the improved MPM gives a correct deformation of \( \nu = 1/3 \).

Fig.8 shows the calculation results of the Poisson’s ratio when the aspect ratio of the rectangular model is selected as a parameter. Here the aspect ratio is defined by the length ratio of two perpendicular sides of a rectangular model (\( y/x \) in Fig.9). This result indicates that the correct deformation can be obtained regardless of the shape of the elements.

Another important feature of the improved MPM is that even though the isotropic material is assumed...
in this paper, it is not mandatory to assume that as is done in eq.(7). We can apply the improved MPM to an anisotropic membrane by just giving an appropriate $D$.

5 Conclusion

The new formulation for the multi-particle model was derived to overcome the difficulties the conventional formulation has had. The new formulation can deal with an arbitrary shape, arbitrary material parameters membrane without so much increasing the calculation cost nor losing the simplicity of formulations. The proposed formulation was validated in terms of a Poisson’s ratio and the shape of an element by some simulations. The resulted algorithm enables the multi-particle mode to be used for an arbitrary-shaped membrane, that is desired for the analysis of the solarsail membrane dynamics.

References
