A New Approach for Spacecraft Rendezvous Control on Elliptic Orbits

Kazuki Hayashi Kazuo Tsuchiya

Kyoto University, Yoshidahonmachi, Sakyouku, Kyoto, Japan

— Abstract —

This paper presents considerations of formulation and design for spacecraft rendezvous control on elliptic orbits around the earth. Spacecraft rendezvous is a key technique for orbital services to the target spacecraft such as resupply, review, assembling or mend. In order to formulate, we need to derive a linearized equation of relative motion between two spacecrafts (target and chaser) on an elliptic orbit. The solution of this equation is generally composed of periodic term and secular term (which is called 'drift motion' in this paper). Based on this fact, we can obtain a discrete state transition law of relative motion, which enables us to realize simple rendezvous control of the chaser.

I. Introduction

Spacecraft rendezvous problem continues to be of interest for about a half century. Recently, detailed studies, experiments in space, and hardware laboratory demonstrations of autonomous rendezvous and docking operation have been actively enforced in many countries. For example, NASDA (presently JAXA) launched the ETS-VII in 1997 to demonstrate autonomous rendezvous and docking mission on a low earth orbit.

Almost of rendezvous and docking missions are enforced on circular orbits (like the ETS-VII), because relative motion of two spacecrafts is easy to analyze, which is represented as the well-known Hill-Clohessy-Wiltshire (HCW) equation. However some missions must be enforced on an elliptic orbit for practical reasons. So rendezvous problem on elliptic orbits is considered in this paper. Also many researchers are interested in formation flying [2][3][4]. The contents or results of this paper will benefit formation flying control.

The ordinary differential equation of relative motion on elliptic orbits is nonhomogeneous, because the gravitational field around spacecrafts is time-varying. The equation of relative motion and its solution are derived in section II and III. The solution is composed of periodic term and secular term. Based on this fact, a discrete state transition law of relative motion with chaser’s impulsive accelerations can be obtained (see the section IV). And in the section V, the effect of linearization error of the equation of motion is investigated. This paper considers two control schemes whose main purpose is “periodic state fixed” or “fuel cost minimization” in the section VI. In the end, simulation results are shown and conclusions of this paper are addressed.
II. Equation of Relative Motion

It is supposed that an uncontrollable satellite called target is flying on an elliptic orbit (eccentricity and semi-major-axis are $e, a$ respectively) around the earth. And a controllable spacecraft called chaser is near the target. An earth-centered inertia (ECI) frame is defined as fig.1 and positions of the target and the chaser on ECI frame are represented as $\mathbf{R}, \mathbf{r}$ respectively. So that equations of motion of the tar-

t get and the chaser are represented as follows:

$$\ddot{\mathbf{R}} = -\frac{\mu \mathbf{R}}{|\mathbf{R}|^3},$$
$$\ddot{\mathbf{r}} - \frac{\mu (\mathbf{R} + \mathbf{r})}{|\mathbf{R} + \mathbf{r}|^3},$$

(1)

where $\mu$ is the gravity constant of the earth and $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$. Now it is assumed that the chaser is enough close to the target in comparison with to the center of the earth, so that an approximation

$$\frac{\mathbf{R} + \mathbf{r}}{|\mathbf{R} + \mathbf{r}|^3} \approx \frac{1}{|\mathbf{R}|^3} \left( \mathbf{R} + \mathbf{r} - \frac{3 \mathbf{R} \cdot \mathbf{r}}{|\mathbf{R}|^2} \right)$$

is derived. Then the linearized equation of relative motion between the target and the chaser can be obtained as

$$\ddot{\mathbf{r}} = -\frac{\mu}{|\mathbf{R}|^3} \left( \frac{\mathbf{r} - \frac{3 \mathbf{R} \cdot \mathbf{r}}{|\mathbf{R}|^2}}{|\mathbf{r}|^3} \right),$$

(2)

Here a target-centered local vertical/local horizontal (LVLH) frame is defined as fig.1, where $z$-axis is vertical (i.e. directed to the earth), $y$-axis is to out-of-plane and $x$-axis is determined to complete the right hand frame. Let $\mathbf{r}$ be the position of the chaser from the target on LVLH frame and $\theta(t)$ be the true anomaly of the target, and then the vector-formed equation of relative motion (2) can be rewritten as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -k\dot{\theta}^2 x + 2\dot{\theta} \ddot{x} + \dot{\theta}^2 x \\ -k\dot{\theta}^2 y \\ 2k\dot{\theta}^2 z - 2\dot{\theta} \ddot{x} - \dot{\theta}^2 z \end{bmatrix},$$

(3)

where $k := \mu/(2h)^2$ is a constant which is determined by the areal velocity of the target $h$. A transformation is defined as

$$[\xi, \eta, \zeta]^T = \rho [x, y, z]^T, \quad \rho := 1 + e \cos \theta,$$

to simplify this equation, and then the eq.(3) becomes

$$\begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta} \end{bmatrix} = \begin{bmatrix} 2\zeta \\ -\eta \\ 3\zeta/\rho - 2\xi' \end{bmatrix}, \quad \rho = \frac{d}{d\theta}.$$

(4)

Remark 1: If the target is on a circular orbit, we can get $e = 0, \theta = nt$ ($n$ is the mean motion). Therefore the eq.(4) becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 2n\dot{z} \\ -n^2 y \\ 3n^2 z - 2n\dot{x} \end{bmatrix},$$

which is the well-known HCW equation. The solution of the HCW eq.1 can be easily obtained as follows:

$$x = K_1 - 2K_2 \cos nt + 2K_3 \sin nt + 3k^2K_4 t,$$
$$y = K_5 \cos nt + K_6 \sin nt,$$
$$z = K_2 \sin nt + K_3 \cos nt + 2K_4,$$

where $K_1, \ldots, K_6$ are integral constants. Note that this solution is composed of periodic term characterized by $K_1, \ldots, K_3$ and secular term $3k^2K_4 t$. In the next section, we will find the solution on an arbitrary elliptic orbit has similar properties to one on a circular orbit.

III. Relative Motion on Elliptic Orbits

The equation of spacecrafts' relative motion (4) is derived in the previous section. This differential equation is nonhomogeneous, except on a circular orbit ($e = 0$). The solution for a time interval $[t_0, t]$ can be derived as follows [1]:

$$\xi = K_1 - K_2(c + \cos \theta) + K_3(s + \sin \theta) + 3K_4\rho^2 J,$$
$$\eta = K_5 \cos \theta + K_6 \sin \theta,$$
$$\zeta = K_2 s + K_3 c + K_4(2 - 3\epsilon s J),$$

(5)

(6)

(7)

where $K_1, \ldots, K_4$ are integral constants determined by the initial state and

$$s = \rho \sin \theta, \quad c = \rho \cos \theta, \quad J = k^2(t - t_0).$$

The in-plane solution (5),(7) can be rewritten in matrix notation as

$$\begin{bmatrix} \xi(t) \\ \eta(t) \\ \zeta(t) \end{bmatrix} = \Phi_{0t}(t) \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix},$$

(8)

where $[\xi, \eta, \zeta]^T, \quad [K_1, K_2, K_3, K_4]^T$ are an in-plane state vector, an integral constant vector respectively, and $\Phi_{0t}(t)$ is

$$\Phi_{0t}(t) = \begin{bmatrix} 1 & -c - \cos \theta & 3 \rho^2 J \\ 0 & s & 2 - 3\epsilon s J \\ 0 & 2s & 2c - e \end{bmatrix}.$$
By LVLH coordinate \( \mathbf{x} = [x, z, \dot{x}, \dot{z}]^T \), the in-plane state transition for \( [t_0, t] \) can be written as

\[
\mathbf{x}(t) = A_{\theta(t)}(\mathbf{x}-\mathbf{\xi}(t)) A_{\theta(t)}^{-1} \mathbf{x}(t_0),
\]

(9)

where

\[
A_{\theta}(\mathbf{x}-\mathbf{\xi}) = \begin{bmatrix}
\rho \rho I_{2 \times 2} - e \sin \theta I_{2 \times 2} & O_{2 \times 2} \\
-e \sin \theta I_{2 \times 2} & I_{2 \times 2} / (\kappa^2 \rho)
\end{bmatrix}
\]

(10)

is a transformation matrix from \( \mathbf{x} \)-coordinate to LVLH coordinate when the true anomaly is \( \theta(t) \).

On the other hand, out-of-plane motion can be expressed in matrix notation as

\[
\begin{bmatrix}
\eta \\
\eta'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
K_5 \\
K_6
\end{bmatrix},
\]

Out-of-plane motion expressed by \( \eta(t) \), \( \eta'(t) \) is a simple harmonic oscillation decoupled from in-plane motion. Because control of out-of-plane motion can be extended to that of in-plane motion, we focus attend only on in-plane motion.

IV. Canonical Transformation

At the first of this section, we define a canonical transformation \( \Phi_{\theta(t)} : \mathbf{x}(\theta(t)) \rightarrow \tilde{\mathbf{K}}(\theta(t)) \) as

\[
\Phi_{\theta(t)} = \Phi_{\theta(t)} A(\mathbf{x}-\mathbf{\xi})^{-1},
\]

\[
\Phi_{\theta(t)} = \begin{bmatrix}
1 & -c & -\cos \theta & \sin \theta & 0 \\
0 & s & c & 2 \\
0 & 2s' & 2c - e & 3 \\
0 & s'' & c' & -3es/\rho^2
\end{bmatrix}^{-1}.
\]

Note that \( \Phi_{\theta(t)}^{-1} \) is similar to \( \Phi_{\theta(t)} \) except for time-dependent terms including \( J \). By this coordinate \( \tilde{\mathbf{K}} \), the state transition law which is equivalent to (9) is represented as

\[
\tilde{\mathbf{K}}(t) = \phi_{\theta(t)} \tilde{\mathbf{K}}(t_0), \quad \phi_{\theta} = \begin{bmatrix}
1 & 0 & 0 & 3J \\
0 & 1 & 0 & -3eJ \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

(11)

It indicates \( \tilde{K}_1, \tilde{K}_2 \) are time-varying variables and \( \tilde{K}_3, \tilde{K}_4 \) are time-invariant variables without control.

Different from LVLH coordinate, \( \tilde{\mathbf{K}} \) has no directly information on relative motion. The following list denotes an interpretation of \( \tilde{\mathbf{K}} \):

\( \tilde{K}_1 \): Offset of relative motion to \( x \)-direction
\( \tilde{K}_2 \): Time-varying part of periodic motion
\( \tilde{K}_3 \): Time-invariant part of periodic motion
\( \tilde{K}_4 \): Secular term for \( \tilde{K}_1, \tilde{K}_2 \) called `drift motion’, or altitude bias of two spacecrafts

V. Effect of Nonlinear Dynamics

The equation of relative motion of two spacecrafts is essentially nonlinear. The original (before linearized) dynamics of relative motion was represented as eq.(1), and the linearized relative motion was represented eq.(11) by \( \tilde{\mathbf{K}} \)-coordinate. For example, the fig.2 indicates the original dynamics of relative motion when the initial state is \( \tilde{K}_1(0) = \tilde{K}_2(0) = \tilde{K}_3(0) = 0, \tilde{K}_4(0) = 300[m] \). Comparing with the

\[
\text{fig. 2: Nonlinearity of Relative Motion (Upper figure is shown } \tilde{K}_1, \tilde{K}_2 \text{ and lower } \tilde{K}_4 \text{)}
\]

linearized dynamics of relative motion, \( \tilde{K}_4 \) is not constant especially on the perigee. This example reveals that nonlinearity of eq.(1) has much effect on relative motion when both spacecrafts are near the perigee. Therefore attentions are needed to control spacecrafts on the perigee.

VI. Planning Trajectory

Elementary Equation for Control Design

Rendezvous control schemes are required convergence of \( \mathbf{x}_q \), which is equivalent to \( \tilde{\mathbf{K}} \rightarrow 0 \). Now we propose a control scheme: the chaser is controlled by impulsive thrusts \( \mathbf{a} = (a_x, a_z) \) on the apogee of each period. Then the relative state transition law from the \( n \)-th to the \( (n+1) \)-th period becomes as follows:

\[
\tilde{\mathbf{K}}(n+1) = \phi_{\tau} \tilde{\mathbf{K}}(n) + B \mathbf{a}(n)
\]

(12)
where $\phi_T, B$ are constant matrices,

$$
\phi_T = \phi_0^T, \quad B = \frac{1}{k^2} \begin{bmatrix}
\frac{3k^2T}{1+e} & 2e(1-e) & \frac{3k^2TQ(\theta_0)}{1-e} \\
\frac{3k^2T}{1+e} & 2e(1-e) & \frac{3k^2TQ(\theta_0)}{1-e} \\
-\frac{1+2-\epsilon}{1+e} & 0 & 0 \\
\end{bmatrix}
$$

and $T$ is the time interval for a period. $\vec{K}_1, \vec{K}_2$ are changed by $a_x$ and drift motion caused by $\vec{K}_4$. In contrast, $\vec{K}_3, \vec{K}_4$ are changed only by $a_x$. However a quantity $2\vec{K}_1 - (1-e)\vec{K}_3$ is invariant for any thrust $a$. It declares the chaser cannot be controlled for rendezvous from any initial relative state by thrusts only on the apogee, so another thrust is needed to control all variables. Therefore $x$ directional thrust $b_x/2$ is applied as a couple when the true anomaly is $\theta = \pi \pm \theta_0$. (see fig.3)

Finally the state transition law of relative motion by $\vec{K}$-coordinate is obtained as follows:

$$
\vec{K}(n+1) = \phi_T \vec{K}(n) + \vec{a} \,,
$$

where $\vec{a} = [a_x, a_z, b_x]^T$ and

$$
\beta = \frac{1}{k^2} \begin{bmatrix}
\frac{3k^2T}{1+e} & 2e(1-e) & \frac{3k^2TQ(\theta_0)}{1-e} \\
\frac{3k^2T}{1+e} & 2e(1-e) & \frac{3k^2TQ(\theta_0)}{1-e} \\
-\frac{1+2-\epsilon}{1+e} & 0 & 0 \\
\end{bmatrix},
$$

$P(\theta_0) = \frac{e\cos^2\theta_0 - 2\cos\theta_0 + e}{(1 - e^2)(1 - e^2)}$,  
$Q(\theta_0) = \frac{1 - e\cos\theta_0}{1 - e^2}$.  

And above equation can be rewritten as follows:

$$
\vec{K}_1(n+1) = \vec{K}_1(n) + D + \Delta ,
$$

$$
D = \begin{bmatrix}
D \\
0 \\
0 \\
\end{bmatrix}, \quad D = 3k^2T\{\vec{K}_4(n) + \Delta_4\},
$$

$$
\Delta = \begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3 \\
\Delta_4 \\
\end{bmatrix}, \quad \Delta_1 = \frac{2e^2a_x}{1+e}, \quad \Delta_2 = \frac{2e^2a_z}{1+e}, \quad \Delta_3 = \frac{2e^2b_x}{1+e} + \frac{P(\theta_0)}{Q(\theta_0)}b_x,
$$

where $D$ is the drift part and $\Delta$ is the directly controllable part.

**Remark 2**: If we decided another thrust point is the perigee, we would need only two thrust points to control the chaser. However because nonlinear effect is great near the perigee as shown in the section V, it is risky to control there.

**Periodic State Fixed**

If the desired relative states of each period is given, the command thrusts and the relative trajectory are automatically decided.

When the desired relative state is given as

$$
\vec{K}_1(n+1) = \lambda \vec{K}_1(n), \quad \vec{K}_2(n+1) = \lambda \vec{K}_2(n), \quad \gamma(n+1) = \gamma(n), \quad \gamma = 2\vec{K}_1 - (1-e)\vec{K}_3 ,
$$

one can get

$$
\Delta_2 = \frac{\Delta_1}{2 - e} = \frac{(1 - \lambda)(e\vec{K}_1 + \vec{K}_2(n))}{1 - e},
$$

$$
\Delta_4 = \vec{K}_4(n) = \frac{(1 - \lambda)(\vec{K}_1(n) + (2 - e)\vec{K}_2(n))}{(1 - e^2)},
$$

$$
\Delta_3 = \frac{(1 - \lambda)(\gamma(n) + 2\Delta_4)}{1 - e},
$$

and then the command thrusts of the chaser in the $n$-th period as follows:

$$
\begin{bmatrix}
\begin{bmatrix}
a_x \\
a_z \\
b_x \\
\end{bmatrix} = \begin{bmatrix}
\frac{P(\theta_0)\Delta_1 - Q(\theta_0)\Delta_3}{2Q(\theta_0) - (1 - e)P(\theta_0)} \\
\frac{\Delta_4}{2Q(\theta_0) - (1 - e)P(\theta_0)} \\
\frac{-\Delta_4}{2Q(\theta_0) - (1 - e)P(\theta_0)} \\
\end{bmatrix}
\end{bmatrix}
$$

**Cost Minimization**

It is often important for spacecraft control to minimize fuel cost. We propose a formulation for optimization in this paragraph. If the initial state $\vec{K}_0$ is given, the state after $N$ periods $\vec{K}_N$ (on the apogee) is obtained as

$$
\vec{K}_N = \phi_T^N \vec{K}_0 + \sum_{n=1}^N \phi_T^{N-n} \vec{a}_n ,
$$

and more simple form

$$
\vec{K}_N = F \vec{u} + \vec{v} ,
$$

where $\vec{u} = [\vec{a}_1, \ldots, \vec{a}_N]^T$, $F = [\phi_T^{N-1}, \ldots, \phi_T, I]$ and $\vec{v} = \phi_T^N \vec{K}_0$. Now we impose a condition that spacecraft rendezvous is complete on the apogee of the $N$-th period. Then we can get a condition

$$
\vec{K}_N = F \vec{u} + \vec{v} = 0 .
$$

The cost function for optimization is given as

$$
J(u) = \|u\|_1 = \sum_{n=1}^N \|\vec{a}_n\|_1 ,
$$
where $\| \bullet \|_1$ is 1-norm defined by the summation of absolute values of elements of the vector $\bullet$. To convert above condition and cost function to the standard form of linear programming (LP), we define slack variables and, we can rewrite above condition and cost function as

$$\begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix} \leq \begin{bmatrix} v + s \\ -v + s \end{bmatrix}$$

$$J(u^+, u^-) = u^+ + u^- \rightarrow \text{minimize}$$

where $u = u^+ - u^-$. Moreover we impose a restriction of acceleration $|\ddot{a}_n| \leq u_{sat}$, so the standard form LP problem can be written as ([3])

$$\begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix} \leq \begin{bmatrix} v + s \\ -v + s \\ u_{sat} \\ u_{sat} \end{bmatrix}.$$  \hfill (22)

**VII. Simulation Results**

One shows simulation results for the proposed rendezvous control schemes in this section. The initial conditions of both simulations are given as

$$\bar{K}_1 = 20{\text{}[\text{km}]}, \bar{K}_2 = \bar{K}_3 = \bar{K}_4 = 100{\text{[m]}},$$  \hfill (23)

and parameters for simulations are shown in the tables (tab.1,tab.2,tab.3).

**tab. 1: Simulation parameters for both schemes**

<table>
<thead>
<tr>
<th>Name</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>eccentricity</td>
<td>$e$</td>
<td>about 0.72</td>
</tr>
<tr>
<td>perigee height</td>
<td>$R_{min}$</td>
<td>500$[\text{km}]$</td>
</tr>
<tr>
<td>apogee height</td>
<td>$R_{max}$</td>
<td>36.000$[\text{km}]$</td>
</tr>
<tr>
<td>initial true anomaly</td>
<td>$\theta(0)$</td>
<td>0$[\text{rad}]$</td>
</tr>
<tr>
<td>simulation time</td>
<td>$*$</td>
<td>12 periods</td>
</tr>
</tbody>
</table>

**tab. 2: Parameter for “Periodic State Fixed” scheme**

| convergence ratio | $\lambda$ | 0.6 |

**tab. 3: Parameters for “Cost minimization” scheme**

| mission interval | $N$ | 10 |
| saturation of thruster | $u_{sat}$ | 2.0$[\text{m/sec}^2]$ |

The fig.4 shows series of homothetic trajectories with a ratio of $\lambda$. It indicates gradual rendezvous is realized. And by the fig.5 and fig.6, we can find controlled trajectory by the “Cost minimization” scheme is in more wide curve than the above trajectory, and its total cost certainly is smaller than that of “Periodic State Fixed” scheme.

**VIII. Conclusion**

A new approach to spacecraft rendezvous control has been addressed in this paper. We can obtain a state transition matrix of the linearized relative equation when it is assume spacecrafts are on an arbitrary elliptic orbit, and then define a useful canonical transformation to express relative motion of two spacecrafts. Relative motion is composed of periodic motion and drift motion. This paper proposed two rendezvous control schemes maximizing the effect of drift motion. And we verify the chaser spacecraft can be controlled to approach to the target by these schemes in simulations.

**References**


Appendix: Derivation of the Solution

The derivation for the solution of the eq.(4) is indicated here. The eq.(4) is written again,

\[ \zeta'' = 2\zeta', \]  
\[ \eta'' = -\eta, \]  
\[ \zeta'' = \frac{3\zeta}{\rho} - 2\zeta'. \]  

The eq.(25) can be easily solved as

\[ \eta = K_{y1} \cos \theta + K_{y2} \sin \theta, \]  
and eq.(24) can be integrated as

\[ \zeta' = 2\zeta + K_{x1}. \]  

Substituting this to eq.(26), we can obtain the following equation:

\[ \zeta'' + \left( 4 - \frac{3}{\rho} \right) \zeta = -2K_{x1} \]  

To solve eq.(29), we look at the homogeneous form of this equation

\[ \zeta'' + \left( 4 - \frac{3}{\rho} \right) \zeta = 0. \]  

Its fundamental solutions are known as follows (see ref.[1]):

\[ \varphi_1 = \rho \sin \theta, \]  
\[ \varphi_2 = \rho \sin \theta \int_0^\theta \frac{d\tau}{\sin^2 \tau \rho(\tau)} = 2e\rho \sin \theta \left\{ \frac{\sin \theta}{\rho^3} - 3e \int_0^\theta \frac{\sin^2 \tau}{\rho^4(\tau)} d\tau \right\} - \frac{\cos \theta}{\rho} \]

where \( \rho(\tau) := 1 + e \cos \tau. \)

Now we define an elliptical integral:

\[ J(\theta) := \int_0^\theta \frac{d\tau}{\rho^2(\tau)}. \]

We can get \( J(\theta) = k^2(t-t_0) \), because of \( d\theta = k^2 \rho^2 dt \).

So we assume a form of the fundamental solution \( \varphi_2 \) as

\[ \varphi_2 = C_1 sJ + c + C_2, \quad s = \rho \sin \theta, \quad c = \rho \cos \theta. \]

Substituting this equation to eq.(30), we can get

\[ \frac{(2C_1/\rho) \cos \theta + 2e + C_2}{(4 - 3/\rho)} \]  
\[ = \frac{(2C_1 + 4eC_2 + 2e^2) \cos \theta + C_2 - 2e}{0}, \]

therefore we can find that \( C_1 = 3e^2, \quad C_2 = -2e \) and rewrite fundamental solutions as

\[ \varphi_1 = s, \quad \varphi_2 = 3e^2 sJ + c - 2e. \]

Moreover we can find the paticular solution of eq.(29), \( \varphi_3 = -K_{x1}c/e. \)

As a result, the general solution of eq.(29) are

\[ \zeta = K_{x1} s + K_{x2}(3e^2 sJ + c - 2e) - K_{x1}c/e \]  
\[ = K_{x1} s + (K_{x2} - K_{x1}/e)c - K_{x2}e(2 - 3esJ). \]

And substituting this solution to eq.(28) and integrating it, we can obtain

\[ \zeta = K_{x2} - K_{x1}(\rho + 1) \cos \theta \]  
\[ + (K_{x2} - K_{x1}/e)(\rho + 1) \sin \theta - 3K_{x2}ep^2J. \]

Finally replacing the integral constants, we can derive the solution (5)-(7).