Design of Halo Orbit around L2 of Earth-Moon System

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Abstract

In Earth-Moon system, L2 point lies behind the Moon as seen from the Earth. By using a satellite around this point, it is possible to communicate with the far side of the Moon. In this study, we show a control method to keep Halo orbit around the L2 point. Then, to evaluate the Moon eccentricity effect, we applied the restricted three body problem and examine the possibility of keeping the orbit.

INTRODUCTION

We don’t have sufficient knowledge about the far side of the Moon as contrasted to near side. One of the biggest reason is that the Moon keep the same face towards the Earth and we can’t observe or communicate with the far side of the Moon from the Earth. If we want to establish communication between them, we need the relay satellite. Orbits around L2 point of Earth-Moon system is applicable to set such satellite. Among five Lagrange points of Earth-Moon system, L2 lies behind the Moon as seen from the Earth. L2 is shielded by the disk of the Moon, but
Halo orbit around this point is visible from the Earth.

This study shows the way to realize the Halo orbit by some control law with a constant thrust and constraining direction in a certain plane. Usual communication satellites must set its antenna to the Earth constantly, so this control law has advantages to design the satellite’s structure. In addition, a control law with a constant thrust has wide application such as a control by ion engine [1].

Derivation of Control Law

In this section, we derive the control law from linear equation of motion in the circular restricted three-body problem. The control law is derived under the constraints that the thrust is constant and the thrust direction rotate at constant rate.

The circular restricted three-body problem in Earth-Moon system is described as Fig.1. Here the long dimension is normalized with the distance from Earth to Moon, and $\mu$ denotes the constant number such that

$$\mu = \frac{M_m}{M_e + M_m}$$  \hspace{1cm} (1)

where $M_e$ and $M_m$ are the mass of Earth and Moon.

Fig.1 The circular restricted three-body problem

The nonlinear equations of motion around L2 are described as

$$\ddot{x} - 2\dot{y} - (x + 1 - \mu + \gamma_L) = -\frac{1 - \mu}{r_1^3}(x + 1 + \gamma_L) - \frac{\mu}{r_2^3}(x + \gamma_L) + a_x$$

$$\ddot{y} + 2\dot{x} - y = -\frac{1 - \mu}{r_1^3}y - \frac{\mu}{r_2^3}y + a_y$$

$$\ddot{z} = -\frac{1 - \mu}{r_1^3}z - \frac{\mu}{r_2^3}z + a_z$$  \hspace{1cm} (2)

where $\mathbf{a} = (a_x, a_y, a_z)$ denotes the thrust acceleration. Here the time scale is normalized with the Moon’s period of revolution. Assuming that the satellite moves around the neighborhood of L2, i.e., $x, y, z$ are small, the linearized equations of motion around the L2 can be described as

$$\ddot{x} - 2\dot{y} - (2B_L + 1)x = a_x$$

$$\ddot{y} + 2\dot{x} + (B_L - 1)y = a_y$$

$$\ddot{z} + B_L z = a_z$$  \hspace{1cm} (3)

where

$$B_L = \frac{1 - \mu}{(1 + \gamma_L)^3} + \frac{\mu}{\gamma_L^3}.$$  \hspace{1cm} (4)

Now, to realize the circular orbit, we consider the constraints as following relation.

$$x = y_0(\beta \cos \theta - \sin \theta) \sin \omega t$$

$$y = y_0 \cos \omega t$$

$$z = y_0(\cos \theta + \beta \sin \theta) \sin \omega t$$  \hspace{1cm} (5)

where $y_0$ is $y$-element of initial position and $\beta$ denotes the amplitude ratio of the out-of-plane motion to the in-plane motion, and $\omega$ denotes the angular velocity of in-plane circular motion. This solution realizes the circular trajectory on the plane perpendicular to $(\cos \theta, 0, \sin \theta)^T$ axis.
Next, we consider the constraints about the acceleration vector as following relation.

\[
\begin{align*}
  a_x &= \alpha \sin \phi \sin \omega t \\
  a_y &= \alpha \cos \omega t \\
  a_z &= -\alpha \cos \phi \sin \omega t
\end{align*}
\]  

(6)

where \(\alpha\) denotes the constant magnitude of \(a\). This constraints implies that the thrust is constant, and it’s direction rotates at a constant rate on the plane perpendicular to \((\cos \phi, 0, \sin \phi)^T\) axis.

Substituting these equation into (3), we obtain the following equation.

\[
y_0(\sin \theta - \beta \cos \theta)\omega^2 + 2y_0\omega + (2B_L + 1)y_0(\sin \theta - \beta \cos \theta) = \alpha \sin \phi
\]

\[
- y_0(\cos \theta + \beta \sin \theta)\omega^2 + B_Ly_0(\cos \theta + \beta \sin \theta) = -\alpha \cos \phi
\]

(7)

Using these equation to eliminate \(\alpha\) and \(\beta\), we get following equation about \(\omega\).

\[
\begin{align*}
  (1 + \cos \theta \cos \phi + \sin \theta \sin \phi) \omega^4 + & \quad 2(\sin \theta + \sin \phi)\omega^3 \\
  + & \quad (1 + B_L + (B_L - 2) \cos \theta \cos \phi - 2B_L \sin \theta \sin \phi) \omega^2 \\
  - & \quad 2B_L (\sin \theta + \sin \phi)\omega \\
  - & \quad \{B_L + 2B_L^2 - (1 + B_L - 2L^2) \cos \theta \cos \phi - (B_L \pm 2B_L^2) \sin \theta \sin \phi\}\omega = 0
\end{align*}
\]  

(8)

In this equation, the coefficient of \(\omega^4\) is opposite sign of the constant term. This implies that \(\omega\) has the solution for any \(\theta\) or \(\phi\). Here, to realize the orbit which is visible from Earth, we think about the case of \(\theta = 2n\pi\) or \(\theta = 2(n - 1)\pi\). For these case, eq.(8) becomes as follow.

\[
\begin{align*}
  \{\omega^4 + (4B_L - 2)\omega^2 + (2B_L + 1)^2\} \beta^2 \\
  = 12\omega B_L \beta + 2\omega^2 + 2B_L - 1 = 0
\end{align*}
\]  

(9)

From eq.(4), we can calculate the specific value of \(B_L\) and obtain the relation between \(\omega\) and \(\beta\) as Fig.2.

![Fig.2 \(\omega - \beta\)](Image)

\(\beta\) can exist for the following range of \(\omega\).

\[
1.5222 < \omega < 1.8189 \quad (10)
\]

Substituting (7) into (6), we get following thrust acceleration.

\[
\begin{align*}
  a_x &= \{2\omega \mp (\omega^2 + 2B_L + 1) \beta\} y_0 \sin \omega t \\
  a_y &= \{-\omega^2 + B_L - 1 \pm 2\omega \beta\} y_0 \cos \omega t \\
  a_z &= \{\mp (\omega^2 - B_L)\} y_0 \sin \omega t
\end{align*}
\]  

(11)

If we set \(a_x = 0\), we can hold the direction of the thrust acceleration in constant angle to the Earth. For this condition, the value of \(\omega\) in no dimensional form is

\[
\omega = 1.5678, 1.8182
\]

and both \(\omega\) satisfy eq.(10).

We design Halo orbit using this \(\omega\), and verify the control law in next section.

**Numerical Simulation**

In this section, we verify the control law by numerical simulation. The control law is applied to the nonlinear
equation of motion of the circular restricted three-body problem in eq.(2). By trial-and-error iterations, we select the initial conditions as Table. 1. Fig.3 and Fig.4 show the orbit of satellite in the simulation.

<table>
<thead>
<tr>
<th>ω</th>
<th>Initial position(km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5678</td>
<td>(−81.9590056, 2500.0, 0, 0)</td>
</tr>
<tr>
<td>1.8182</td>
<td>(−80.9770755, 2500.0, 0, 0)</td>
</tr>
</tbody>
</table>

**Fig.3 Nonlinear Simulation at ω = 1.5678**

In Fig.3, L2 shifts for approximately 81.96km to barycenter. The direction of the thrust acceleration is rotated around x-axis as same as radial direction of circular orbits in y−z plane.

**Fig.4 Nonlinear Simulation at ω = 1.8182**

In Fig.4, L2 shifts for approximately 80.98km to barycenter. The direction of the thrust acceleration is rotated around x-axis against the orbital rotation in y−z plane. In any case, the direction of the thrust acceleration holds in constant angle to the Earth.

**Restricted Three-Body Problem**

In the Earth-Moon system, we should think about the effect of the eccentricity of the Moon’s orbit as well as nonlinear effect[2]. In this section, we applies the control law to the nonlinear equation of motion of the elliptic restricted three-body problem. The nonlinear equations of motion around the barycenter of Earth and Moon are described as

\[
\frac{d^2x}{df^2} - \frac{2}{df} \frac{dy}{df} = \frac{1}{1+e \cos f} \left\{ x - \frac{1}{r_1^2} (x + \mu) - \frac{\mu}{r_2^2} (x - (\mu - 1)) + \alpha_x \right\}
\]

\[
\frac{d^2y}{df^2} + \frac{2}{df} \frac{dx}{df} = \frac{1}{1+e \cos f} \left\{ y - \frac{1}{r_1^2} y - \frac{\mu}{r_2^2} y \right\} + \alpha_y
\]

\[
\frac{d^2z}{df^2} = \frac{1}{1+e \cos f} \left\{ -ze \cos f - \frac{1}{r_1^2} z - \frac{\mu}{r_2^2} z \right\} + \alpha_z
\]

(12)

where e and f denotes the eccentricity and true anomaly of the moon’s orbit and the long dimension is normalized with the instantaneous distance from Earth to Moon. The equilibrium point, i.e., L2 point derived from these equation can be expressed as in the case of circular restricted three-body problem.
using the instantaneous distance from Earth to Moon. So we use the coordinate around this instantaneous equilibrium point, and the control law using \( f \) instead of \( t \).

By the change of coordinate, the nonlinear equations of motion around the instantaneous L2 are described as

\[
\begin{align*}
\frac{d^2 x}{df^2} - 2 \frac{dy}{df} &+ \frac{1}{1+e \cos f} (x+1-\mu+\gamma_L) \\
&= \frac{1}{1+e \cos f} \left\{-\frac{1-\mu}{r_1}(x+1)+\gamma_L-\frac{\mu}{r_2}(x+\gamma_L)\right\}+a_x \\
\frac{d^2 y}{df^2} + 2 \frac{dx}{df} &+ \frac{1}{1+e \cos f} \left\{y-\frac{1-\mu}{r_1}y-\frac{\mu}{r_2}y\right\}+a_y \\
\frac{d^2 z}{df^2} &+ \frac{1}{1+e \cos f} \left\{-2 \mu \cos f - \frac{1-\mu}{r_1}z-\frac{\mu}{r_2}z\right\}+a_z
\end{align*}
\]

We applied the control law for these equation, and select the initial position by trial-and-error iterations as \((-83.95308753, 2500.0, 0.0)\) for \( \omega = 1.5678 \). Fig.5 show the orbit motion of the satellite under eq.(13). The radius of the orbit varies in proportion to the instantaneous distance from Earth to Moon.

**CONCLUSION**

In this study, we show the way to realize the Halo orbit around L2 in Earth-Moon system. The control law is derived from circular restricted three-body problem under the constraints of the constant thrust and the direction of the thrust acceleration. This is very useful for many types of satellite.

This control law is also applied to the elliptic restricted three-body problem, and we see that the Halo orbit holds even where the eccentricity of the Moon’s orbit affects it’s equilibrium point.

**REFERENCES**
