Earth Escape Trajectories Starting from L2 Point

Masaki Nakamiya (Sokendai University), Hiroshi Yamakawa (JAXA)

Abstract

The L1 and L2 points of the Sun-Earth system, which are located about 1.5 million km away from the Earth in the sun and anti-sun direction, are widely used as suitable locations for space observatories to a great extent. These points are also thought as a gateway for the interplanetary transfer in the near future. This paper investigates the escape trajectories from the Earth’s gravity leaving the Sun-Earth L2 point in the restricted three-body problem. Firstly, we discussed the one-impulse escape strategy assuming velocity correction only at L2 point, and the two-impulse escape strategy assuming the first velocity correction at L2 point and the second velocity correction at the subsequent perigee point. By keeping the hyperbolic excess velocity constant, the required total velocity correction amount and the flight path geometry are analyzed both analytically using Jacobi integral. Finally, optimal escape trajectories of these strategies are calculated by SQP (Sequential Quadratic Programming method), optimal multi-impulse escape strategy is also investigated.

1. INTRODUCTION

These days, due to increasing interest in scientific exploration of planets and asteroids, optimal trajectories to these celestial bodies have been investigated extensively. In general, these explorations are in need of a large amount of fuel for spacecraft, and various studies pursuing low energy transfer trajectories have been presented. Although the earth escape starting from the low earth orbit was thoroughly investigated in Ref. 7, the escape from the Earth’s sphere influence starting from L1 and L2 point is not yet fully understood.

The L1 and L2 points of the Sun-Earth system are good positions for space observatories because an object around L1 and L2 points can maintain the same orientation with respect to Sun and Earth. Several astronomy satellites have then already utilized these points i.e., Halo/Lissajous orbits. In addition, transfer to the interior and exterior region of the earth is relatively easy by addition of a little energy at L1 and L2 points. Therefore, these points are also considered as a candidate of gateway for the interplanetary transfer in the near future (Refs. 8 and 9).

For these reasons, the purpose of this paper is to investigate the earth escape trajectories leaving the Sun-Earth L2 point, which may well be considered as the initial departure sequence of the interplanetary transfer. The planar circular restricted Sun-Earth-spacecraft (S/C) three-body problem is assumed for the Sun-Earth gravity model. The total required velocity correction amount, the time of flight and the flight path geometry are investigated. Since we can treat the dynamics starting from Sun-Earth L1 point similarly to that of L2, we concentrate on the orbital dynamics starting from L2 point in this paper.

Matsumoto et al. examined the low-thrust earth escape from Sun-Earth L2 point using Electric Delta-V Earth Gravity Assist (EDVEGA), based on the result of impulsive escape maneuvers by fixing the total required velocity correction amount. While, this paper fixes the earth escape energy (C3) and minimizes the total velocity correction amount. It is because the earth escape energy (C3) and the escape direction from the Earth are mainly determined by the interplanetary path design. This study discusses firstly two earth escape strategies. One is the one-impulse escape that is the direct escape with \( \Delta V_1 \) only at L2, and the other is the two-impulse escape assuming \( \Delta V_1 \) at L2 point and \( \Delta V_2 \) at the subsequent perigee point. These two strategies are analyzed using the Jacobi integral to estimate the minimum required velocity correction, and then are optimized numerically to verify the analytical estimation of the required velocity correction and to exemplify the geometry of the escape trajectories. Finally, the multi-impulse escape strategy is discussed.

2. DYNAMIC MODEL

The escape from the Earth’s gravity starting from the L2 point is investigated assuming the planar circular restricted Sun-Earth-S/C three-body model (Fig. 1). Let \( m_1 \) and \( m_2 \) be the masses of Sun and Earth respectively. The origin of the rotating frame system is located at their mutual center of mass, and x-axis is the line from Sun to Earth. The total mass of \( m_1 \) and \( m_2 \), the angular velocity and the distance from Sun to Earth are normalized to one, and the coordinates of the Sun and the Earth are (-\( \mu \), 0) and (1-\( \mu \), 0), where \( \mu = m_2/(m_1+m_2) \).

The equations of motion for S/C are given by

\[
\begin{align*}
\ddot{x} - 2\dot{y} + x = -\frac{\partial U}{\partial x}, \quad \ddot{y} + 2\dot{x} - y = -\frac{\partial U}{\partial y}
\end{align*}
\]

where \( U \) is a potential function as follow.

\[
U = -\frac{1}{d_S} - \frac{\mu}{d_E}
\]

The following equation defines the Jacobi integral, which is a conservative quantity determined from the initial conditions,

\[
J = -(\dot{x}^2 + \dot{y}^2) + x^2 + y^2 + 2\left(\frac{1-\mu}{d_S} + \frac{\mu}{d_E}\right)
\]

![Fig. 1 Sun-Earth line fixed rotating frame.](image-url)
In the above-mentioned model, there are five points where solar gravity, Earth gravity and centrifugal force acting on S/C are balanced. They are called libration points. The L2 point of the Sun-Earth system is located about 1.5 million km away from the Earth in the anti-sun direction. In this study, this L2 point is set as the departure point of S/C.

3. ASSUMPTIONS

In this paper, the radius of Earth’s gravitational sphere is assumed to be 2.0 \times 10^6 km beyond the L2 point, and the hyperbolic excess velocity \( v_{e,1} \) is defined at this distance.

Next, the maximum flight time from libration point L2 to the boundary of the Earth’s gravitational sphere is defined as one year here to avoid extremely long flight time.

Additionally, we assume that the required escape C3 is 1.0 km/s^2 defined at the boundary of the Earth’s gravitational sphere in consideration of transfer to the near earth asteroids with chemical and/or the low thrust propulsion system and transfer to planets with low thrust propulsion system. The cases with C3 other than 1.0 km/s^2 were also studied but omitted here for brevity. The C3 is assumed to be constant throughout the mission.

In this section, the one-impulse escape strategy is analyzed with Jacobi Integral. Fig. 2 illustrates the one-impulse escape strategy model. The S/C starts from L2 point at an angle of \( \theta_{L2} \) by performing the velocity increment \( \Delta V_i \) at L2. And the S/C escapes from the Earth’s gravity directly. The escape velocity is defined as \( V_{ei} \) at the Earth’s gravitational sphere with an angle \( \theta_{ei} \) in the rotating system, where x-axis is defined as zero degree.

Let \( J_{L2} \) and \( J_{AV1} \) be the Jacobi integral values at L2 with S/C initial state (i.e., zero velocity) and after \( \Delta V_i \) have been applied, respectively. Note that the S/C position does not vary during the impulsive maneuver; Eq. (4) yields

\[
J_{AV1} = J_{L2} - \Delta V_i^2
\]

And the escape velocity \( V_{ei} \) is expressed with the Jacobi Integral (Eq. 4) as follows,

\[
v_{ei} = \sqrt{x_{ei}^2 + y_{ei}^2 - J_{AV1} + (x_{ei}^2 + y_{ei}^2) + 2 \left( \frac{1 - \mu}{d_{S,ei}^2} + \frac{\mu}{d_{E,ei}^2} \right)}
\]

where with simple geometrical considerations (cosine formula) it is found that

\[
x_{ei}^2 + y_{ei}^2 = (1 - \mu)^2 + d_{S,ei}^2 - 2d_{S,ei}(1 - \mu)d_{E,ei}\cos(\pi - \theta_{ei})
\]

At this point, C3 = 1.0 km/s^2 is obtained at the Earth’s gravitational sphere by increasing the S/C velocity \( V_{ei} \) with the addition of \( \Delta V_i \) at L2. The total velocity correction \( \Delta V_{total} \) in the one-impulse escape strategy is simply expressed as

\[
\Delta V_{total} = \Delta V_i
\]

4. ESCAPE APPROACH

Jacobi Integral Analysis for One-Impulse Escape

In this section, the one-impulse escape strategy is analyzed with Jacobi Integral. Fig. 3 illustrates the one-impulse escape strategy model. The S/C starts form L2 point at an angle of \( \theta_{L2} \) by performing the velocity increment \( \Delta V_i \) at L2. And the S/C escapes from the Earth’s gravity directly. The escape velocity is defined as \( V_{ei} \) at the Earth’s gravitational sphere with an angle \( \theta_{ei} \) in the rotating system, where x-axis is defined as zero degree.

Let \( J_{L2} \) and \( J_{AV1} \) be the Jacobi integral values at L2 with S/C initial state (i.e., zero velocity) and after \( \Delta V_i \) have been applied, respectively. Note that the S/C position does not vary during the impulsive maneuver; Eq. (4) yields

\[
J_{AV1} = J_{L2} - \Delta V_i^2
\]

And the escape velocity \( V_{ei} \) is expressed with the Jacobi Integral (Eq. 4) as follows,

\[
v_{ei} = \sqrt{x_{ei}^2 + y_{ei}^2 - J_{AV1} + (x_{ei}^2 + y_{ei}^2) + 2 \left( \frac{1 - \mu}{d_{S,ei}^2} + \frac{\mu}{d_{E,ei}^2} \right)}
\]

where with simple geometrical considerations (cosine formula) it is found that

\[
x_{ei}^2 + y_{ei}^2 = (1 - \mu)^2 + d_{S,ei}^2 - 2d_{S,ei}(1 - \mu)d_{E,ei}\cos(\pi - \theta_{ei})
\]

At this point, C3 = 1.0 km/s^2 is obtained at the Earth’s gravitational sphere by increasing the S/C velocity \( V_{ei} \) with the addition of \( \Delta V_i \) at L2. The total velocity correction \( \Delta V_{total} \) in the one-impulse escape strategy is simply expressed as

\[
\Delta V_{total} = \Delta V_i
\]
Optimal Escape Trajectories of One-Impulse Escape Strategy

Next, the escape trajectory of one-impulse escape strategy is optimized by SQP (Sequential Quadratic Programming method) in consideration of the following factors.

- Objective function: \( J = |\Delta V_1| \rightarrow \min \)
- Parameter: magnitude of first velocity correction \( |\Delta V_1| \), departure angle \( \theta_L \)
- Constraint condition: \( C_3 = 1.0 \text{ km}^2/\text{s}^2 \), escape position \( \theta_e \)

The results of the optimal escape trajectories by SQP are shown in Figs. 4 and 5. Fig. 4 shows the examples of optimal escape trajectory. Fig. 6 shows the optimal minimum required \( \Delta V_{\text{total}} \) by SQP and the analytically derived \( \Delta V_{\text{total}} \) by the Jacobi Integral from Eqs. (5) through (11) at each escape position. It is found that the optimal minimum required \( \Delta V_{\text{total}} \) is smallest when the escape position is around 20 degree. Also, the minimum required \( \Delta V_{\text{total}} \) is changing rapidly around 90 degree because the escape velocity direction (the escape trajectory) is changed whether S/C comes close to the Earth once or not (i.e. directly escape). And then, Fig. 6 shows the relation between required \( \Delta V_{\text{total}} \) and the escape velocity direction. It is found that it can escape from Earth’s gravitational sphere to all direction.

Jacobi Integral Analysis for Two-Impulse Escape

In this section, the two-impulse escape strategy is investigated with Jacobi Integral. The S/C deviates from L2 point by adding the velocity correction \( \Delta V_1 \) at L2, and then S/C reaches some subsequent perigee point where the second velocity correction \( \Delta V_2 \) is carried out to escape from the Earth’s gravity.

Fig. 7 illustrates the initial phase that S/C departs from L2 at an angle of \( \theta_L \) with \( \Delta V_1 \) and then arrives at some subsequent perigee point at an angle of \( \theta_p \), where x-axis is defined as zero degree and the arrow of angle indicates the positive direction. Let \( J_{L2} \) and \( J_{\Delta V1} \) be the Jacobi integral values at L2 with S/C initial state (i.e., zero velocity) and after \( \Delta V_1 \) have been applied, respectively. Note that the S/C position does not vary during the impulsive maneuver; Eq. (4) yields

\[
J_{\Delta V1} = J_{L2} - \Delta V_1^2 \quad (12)
\]

When Eq. (4) is used, the velocity at some subsequent perigee point \( v_p \) is expressed as below.
\[ v_{p2} = \sqrt{-J_{AV1} + (x_{p2}^2 + y_{p2}^2) + 2 \left( \frac{1-\mu}{d_{E,p2}} + \frac{\mu}{d_{E,p2}} \right)} \]  
(13)

where with cosine formulation it is found that

\[ x_{p2}^2 + y_{p2}^2 = (1-\mu)^2 + d_{E,p2}^2 - 2(1-\mu)d_{E,p2}\cos(\pi - \theta_{p2}) \]  
(14)

\[ d_{E,p2} = \sqrt{1^2 + d_{E,e2}^2 - 2d_{E,e2}\cos(\pi - \theta_{e2})} \]  
(15)

Next, Fig. 8 illustrates the next step that \( \Delta V_2 \) is applied in the velocity direction at some subsequent perigee point to escape from the Earth’s gravity. A large energy can be efficiently obtained with small \( \Delta V_2 \) at the near earth region. Let \( J_{AV2} \) be Jacobi Integral value after \( \Delta V_2 \) at some subsequent perigee point. Eq. (16) is obtained from Eq. (4):

\[ J_{AV2} = J_{AV1} + v_{p2}^2 - (\Delta V_2 + v_{p2})^2 \]  
(16)

In analogy with Eq. (13), the escape velocity at the boundary of Earth’s gravitational sphere \( v_{e2} \) is obtained

\[ v_{e2} = \sqrt{x_{e2}^2 + y_{e2}^2} = \sqrt{-J_{AV2} + (x_{e2}^2 + y_{e2}^2) + 2 \left( \frac{1-\mu}{d_{e2}} + \frac{\mu}{d_{e2}} \right)} \]  
(17)

where with cosine formula it is found that

\[ x_{e2}^2 + y_{e2}^2 = (1-\mu)^2 + d_{e2}^2 - 2(1-\mu)d_{e2}\cos(\pi - \theta_{e2}) \]  
(18)

\[ d_{e2} = \sqrt{1^2 + d_{e,e2}^2 - 2d_{e,e2}\cos(\pi - \theta_{e2})} \]  
(19)

\( C3 = 1.0 \text{ km}^2/\text{s}^2 \) condition at the boundary of Earth gravitational sphere is obtained by increasing the S/C velocity \( v_{e2} \) with \( \Delta V_1 \) and \( \Delta V_2 \). The total velocity variation \( \Delta V_{total} \) is \( \Delta V_{total} = \Delta V_1 + \Delta V_2 \) 
(20)

From Eqs. (5), (6) and (12) through (20), it is found that \( \Delta V_{total} \) is determined by the following factors;

1) first velocity correction \( \Delta V_1 \)
2) perigee position \( \theta_2 \)
3) perigee altitude \( d_{E,p2} \)
4) escape velocity direction \( \phi_{e2} \)
5) escape position \( \phi_{e2} \)

In the following, we discuss the characteristics of two-impulse escape strategy focusing on the two factors: \( \Delta V_1 \) and \( d_{E,p2} \). The remaining three factors were also investigated but not shown for brevity.

Fig. 9 shows the analytically derived \( \Delta V_{total} \) plotted against \( \Delta V_1 \). It is found that \( \Delta V_{total} \) decreases with decreasing \( \Delta V_1 \). It is because that S/C can approach the near earth region even if \( \Delta V_1 \) is infinitesimal and the earth escape is attained by performing small \( \Delta V_2 \) at that perigee point. Note that \( \Delta V_1 = 0.74 \text{ km/s} \) agrees with the analytically derived \( \Delta V_{total} \) value using one-impulse escape strategy when the escape position is 0 deg and the escape velocity direction is -90 deg (see Fig. 5), when the two-impulse escape strategy is degenerated into the one-impulse escape strategy.

Fig. 10 shows the relation between analytically derived \( \Delta V_{total} \) and \( \Delta V_1 \) (the escape position is 0 deg and the escape velocity direction is -90 deg).

Fig. 11 shows the relation between analytically derived \( \Delta V_{total} \) and the perigee altitude of \( \Delta V_2 \). The perigee position \( \theta_{p2} \) as well as the magnitude \( \Delta V_2 \) at L2 point are changed which cause a variation of the analytically derived \( \Delta V_{total} \). These \( \Delta V_{total} \) decrease with decreasing the perigee altitude. It is because that a large energy can be efficiently obtained with small \( \Delta V_2 \) if the perigee altitude is low. Moreover, this figure indicates that the value of
$\Delta V_{total}$ become larger than 0.74 km/s depending on the perigee position, when the perigee altitude is higher than one million km. This means $\Delta V_{total}$ of two-impulse escape possibly becomes larger than that of one-impulse escape when the perigee altitude is higher than one million km. Therefore, it can be shown that two-impulse escape strategy is effective when leaving L2 point with infinitesimal $\Delta V_1$ and performing $\Delta V_2$ at a low altitude.

**Optimal Escape Trajectories of Two-Impulse Escape Strategy**

In this section, the escape trajectory of two-impulse escape strategy is also optimized by SQP in consideration of the following factors.

- **Objective function:**
  \[ J = |\Delta V_1| + |\Delta V_2| \rightarrow \min \]

- **Parameter:** departure angle $\theta_{e1}$, magnitude of first and second velocity correction $|\Delta V_1|$, $|\Delta V_2|$

- **Constraint condition:** $C3 = 1.0 \text{ km}^2/\text{s}^2$, escape position $\theta_{e1}$

The optimal results of the two-impulse escape trajectory by SQP are shown in Figs. 11 through 13. Fig. 11 shows the example of the optimal two-impulse earth escape trajectory with small $\Delta V_1 = 0.08 \text{ km/s}$ and $\Delta V_2 = 0.42 \text{ km/s}$ at the fourth perigee point where the perigee altitude ($d_{E,p2} = 190,000 \text{ km}$) is the lowest within one-year flight time. Fig. 12 shows the analytically derived $\Delta V_{total}$ by the Jacobi Integral and presents the optimal minimum required $\Delta V_{total}$ by SQP at only several escape positions which area are converged by SQP. We speculate that it can escape comparatively smaller $\Delta V_{total}$ at these areas. Fig. 13 shows the relation between required $\Delta V_{total}$ and the escape velocity direction.

**Optimal Escape Trajectories of Multi-Impulse Escape Strategy**

Moreover, the escape trajectory of multi-impulse escape strategy is optimized by SQP in consideration of the following factors.
Objective function:
\[ J = \sum_{i=1}^{N} |\Delta V_i| \rightarrow \min \] (\(N\) and flight time depend on the escape trajectory shape)

Parameter:
- departure angle \(\theta_{l2}\), magnitude of each velocity correction \(|\Delta V_i| (i = 1, 2, \cdots, N)\)

Constraint condition: \(C_3 = 1.0 \text{ km/s}^2\), escape position \(\theta_{eq}, |\Delta V_i| < 2.0 \text{ km/s} (i = 1, 2, \cdots, N)\)

The optimal results of the two-impulse escape trajectory by SQP are shown in Figs. 14 through 17.

Fig. 14 shows the examples of multi-impulse direct escape trajectory (Type-A) and Fig. 15 shows the relation between \(\Delta V_i\) and the flight time in this case. It is found that \(\Delta V\) are performed at only the vicinity of L2 point, multi-impulse escape strategy is degenerating to one-impulse escape strategy.

Fig. 16 shows the examples of multi-impulse escape trajectory which approaches the earth once (Type-B). And Fig. 17 shows the relation between \(\Delta V_i\) and the flight time in this case. \(\Delta V\) are performed at the vicinity of L2 point and the earth, then it can be said multi-impulse escape strategy is almost degenerating to two-impulse escape strategy. Moreover, we are also under examination about other multi-impulse escape trajectory types now.

Comparison of numerical one-impulse, two-impulse escape strategies and multi-impulse results

The one-impulse, the two-impulse escape strategies and the multi-impulse escape strategy are compared in Figs. 18 and 19. Fig. 18 shows that the flight time of one-impulse escape takes less than 50 days, and that of two-impulse escape takes from 70 to 360 days, which is longer than that of one-impulse escape because S/C comes close to the Earth with \(\Delta V\) once and escapes from the Earth’s gravity. And the flight time of multi-impulse escape depends on the type of trajectory. As mentioned earlier, type-A and type-B are degenerating to one-impulse and two-impulse escape strategy respectively. Then, the flight time of type-A and one-impulse escapes strategy is almost the same value in case of the escape velocity direction = 0 deg, and we can speculate the flight time of type-B and of two-impulse escapes strategy is almost the same value in case of escape velocity direction = -160 deg although the flight time of two-impulse escapes strategy is not obtained.

Fig. 19 shows the comparison of \(\Delta V_{total}\) among the one-impulse, the two-impulse and the multi-impulse escape strategies. The \(\Delta V_{total}\) of multi-impulse escape strategy should be minimum value, one-impulse escape strategy can be presumed the most effective strategy for \(\Delta V_{total}\) when the escape velocity direction is around 0 degree. The two-
impulse strategy might be also the best strategy for $\Delta V_{\text{total}}$ when the escape velocity direction is around $-160$ degree. The optimal escape strategy for $\Delta V_{\text{total}}$ might depend on the escape velocity direction.

The optimal escape strategy for $\Delta V_{\text{total}}$ might degenerate to one-impulse or two-impulse escape strategy, the optimal escape strategy for required total velocity correction can be presumed different with the escape velocity direction. The analysis provides the required velocity correction and the escape direction from the Earth, given the earth escape energy (C3), and can be applied for the design of the interplanetary transfer trajectory. An extension to the other type multi-impulse strategy and the use of Halo orbit instead of L2 point for the starting condition are planned for the future work.

REFERENCES