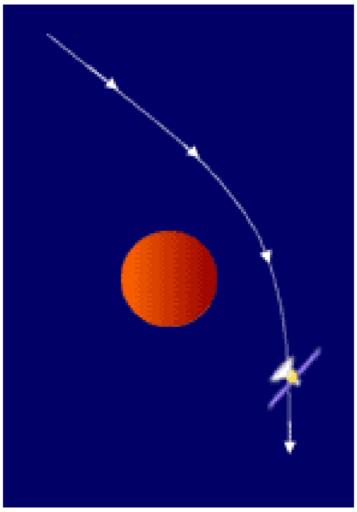
Influence of the Finite Duration on a Fly-by



by
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Abstract

Patched conics is the standard tool for designing fly-by or gravity assist manoeuvres performed by a spacecraft about a planet. In this procedure, the trajectory of the spacecraft about the planet is taken as a perfect hyperbola once the spacecraft is in the sphere of influence of the planet. All other perturbations are neglected. As a consequence the outgoing V∞ is identical to the incoming V∞. In reality, the spacecraft spends a finite time in the sphere of influence of the planet. A typical time is a week. The small perturbations caused by the Sun will affect the ideal hyperbolic trajectory. When the fly-by planet is the Earth, the Moon and J2 cause also small changes in the hyperbola. The small changes in V∞ are the subject of this paper. The perturbation of the Sun can be modelled as a restricted three body problem or analyzed with the standard averaging technique applied to an hyperbolic orbit. We compare our results to the real data of the fly-by past the Earth on 4 March 2005 by the Rosetta spacecraft on his way to the comet Churymov-Gerasimenko (2014).

Abreviations and Notations

2BP : Two Body Problem
3BP : Three Body Problem
COM: Center of Mass
EOM : Equations of motion
SOI : Sphere of Influence
v or $\bar{v}$: any vector. It's modulus is v or $|\bar{v}|$

Patched Conics

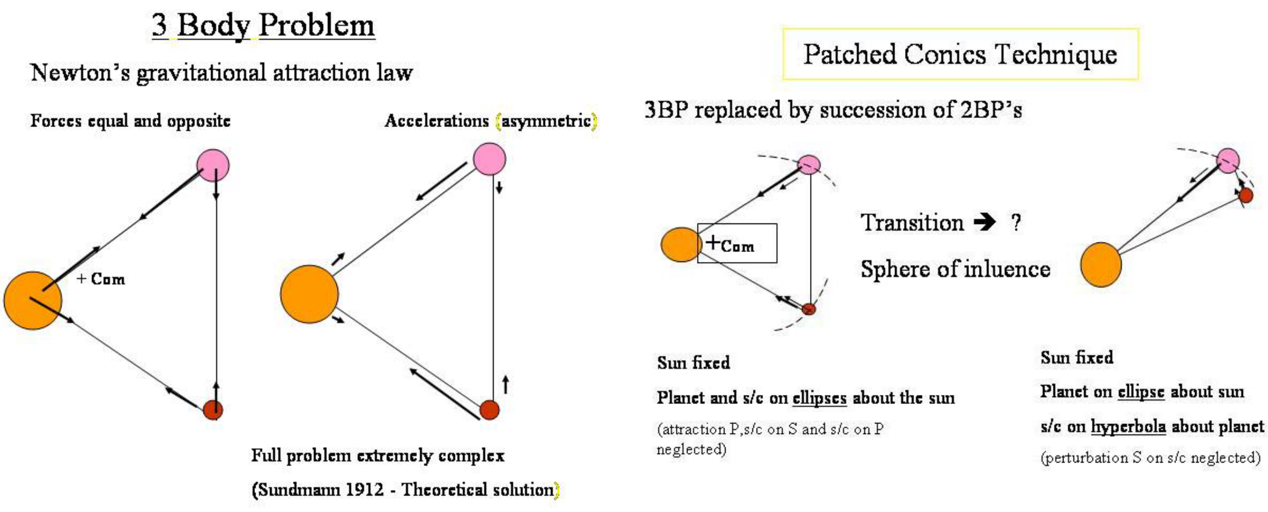

Fig.1 - 3 BP Problem and patched conics

The "patched conics" technique is the standard tool to design fly-by's or gravity assist manoevers. When a spacecraft is in orbit about the sun and approaches a planet, we have, in principle , a 3 Body Problem. In the patched conics technique, the 3BP is replaced by a succession of three 2BP's. A good exposition of this tool is given in **[Battin]**.

Phase 1: spacecraft attracted by the Sun
Phase 2: spacecraft attracted by the planet
Phase 3 : spacecraft attracted by the Sun

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Phase 2 lasts as long as the spacecraft is inside the "Sphere of influence" of the planet.

Energy in the 3BP

The mechanism of gravity-assist is often explained in terms of energy. Therefore we refresh the meaning of energy in the 3BP when the COM of the system is an inertial point. The total energy (E) of a system of 3 particles under Newtonian attraction is the sum of the kinetic (K) and potential (P) energy : E = K + P and

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 - \frac{Gm_1 m_2}{|\bar{r}_2 - \bar{r}_1|} - \frac{Gm_1 m_3}{|\bar{r}_3 - \bar{r}_1|} - \frac{Gm_3 m_2}{|\bar{r}_2 - \bar{r}_3|} \qquad \text{(E1)}$$

How to define the energy E_i of a single particle in : $E = E_1 + E_2 + E_3$ where $E_i = K_i + P_i$

The split up of the kinetic energy is obvious as each term is associated with a single particle. As each term of the potential energy contains two of the three particles, the split up of the potential energy between each of the 3 particles is not obvious. Therefore we start from the Kepler problem[2], where the center of attraction (mass M) is fixed or blocked by constraint forces annihilating the attraction of a single particle m :

$$E_K = -m \frac{\mu}{2a} = m\left(\frac{1}{2}\mathrm{v}^2 - \frac{\mu}{r}\right) = K_1 + P_1 \quad \mu = GM \qquad \text{(E2)}$$

In the 2BP we have :

$$E = \frac{1}{2}m_1 \mathrm{v}_1^2 + \frac{1}{2}m_2 \mathrm{v}_2^2 - \frac{Gm_1 m_2}{|\bar{r}_2 - \bar{r}_1|} \quad = K_1 + K_2 + P \qquad \text{(E3)}$$

We have already here a problem of how to split up P as $P_1 + P_2$. One possibility is to attribute half of the potential energy to each particle:

$$P_1 = P_2 = -\frac{1}{2}\frac{Gm_1 m_2}{|\bar{r}_2 - \bar{r}_1|}$$

In that way , we obtain the correct equations of motion. With 2 arbitrary numbers p,q with p+q = 1 we have the same result. The correct values for p,q follow from the solution of the 2BP. The standard solution of the 2BP is in terms of the relative motion of particle 2 about particle 1and the energy is written as :

$$E = \frac{m_1 m_2}{m_1 + m_2}\left(\frac{1}{2}\mathrm{v}^2 - \frac{G(m_1 + m_2)}{|\bar{r}|}\right) \quad \bar{v} = \bar{v}_2 - \bar{v}_1 ; \bar{r} = \bar{r}_2 - \bar{r}_1 \qquad \text{(E4)}$$

The result is an expression similar to the Kepler problem except for the combination of the 2 masses that appears as a factor.

When the COM is at rest we have : $\sum_{i=1}^{2} m_i \bar{r}_i = \sum_{i=1}^{2} m_i \dot{\bar{r}}_i = \sum_{i=1}^{2} m_i \ddot{\bar{r}}_i = 0$ (E5)

and the equation of motion of particle 1 can be written in terms of $\bar{v}_1$ and $\bar{r}_1$ alone :

$$m_1 \ddot{\bar{r}}_1 = \frac{Gm_1 m_2 \bar{r}}{r^3} = -\left(\frac{m_2}{m}\right)^2 \frac{Gm_1 m_2 \bar{r}_1}{r_1^3} \quad where\ r_i = |\bar{r}_i| \quad and\ m = m_1 + m_2 \qquad \text{(E6)}$$

After multiplication by $\bar{v}_1 = \dot{\bar{r}}_1$ one obtains the energy integral for particle 1:

$$E_1 = \frac{1}{2}\mathrm{m}_1 \mathrm{v}_1^2 - \left(\frac{m_2}{m}\right)^2 \frac{Gm_1 m_2}{r_1} \quad or\ E_1 = m_1\left(\frac{1}{2}\mathrm{v}_1^2 - \frac{m_2}{m}\frac{Gm_2}{r}\right) \qquad \text{(E7)}$$

[2] Mostly introduced as an approximation when body M >> m

The factors p,q for particle 1,2 are respectively m_2 / m and m_1 / m. With these values for p,q the energy of each particle is constant. We have E = E_1+ E_2 and instead of E=E_1(t) + E_2(t).

For the 3BP, we do not know if there is , in general, a good definition for the split up of the potential energy over the 3 particles to obtain

$$E = E_1(t) + E_2(2) + E_3(t) .$$

For the equilateral solutions where each of the particles describes a conic section such that sides of their triangle remain equal, each particle has constant energy :

$$E_i = m_i\left(\frac{1}{2}v_i^2 - \frac{\mu'_i}{r_i}\right) \quad \text{where } \mu'_i = Gm\, w_i^3 \quad \text{and } w_i^2 = \frac{m_j^2 + m_k^2 + m_j m_k}{m^2}$$

but this is the exception rather as the rule and also not valid for all periodic solutions. In the fig-8 periodic solution for 3 equal masses [**Moore 1993**], each of the particles passes in turn through the COM making the potential energy of the particle infinite at that instant. So any explanation of the mechanism of a gravity assist in terms of energy is based on approximations of the full 3BP.

Energy exchange during a flyby

In the 3BP the total energy for the Sun (S), planet (P) and spacecraft or vehicle (V) is :

$$E = \frac{1}{2}m_S v_S^2 + \frac{1}{2}m_P v_P^2 + \frac{1}{2}m_V v_V^2 - \frac{Gm_S m_P}{|\bar{r}_P - \bar{r}_S|} - \frac{Gm_S m_V}{|\bar{r}_V - \bar{r}_S|} - \frac{Gm_V m_P}{|\bar{r}_P - \bar{r}_V|} \qquad \text{(E5)}$$

The standard approximation, for energy considerations is to take the centre of the Sun as origin and neglect its motion about the barycentre. Also, outside the sphere of influence, the potential energy between the Vehicle and Planet are neglected. Eqn.(E.5) reduces then to :

$$E_{approx} = 0 + \frac{1}{2}m_P v_P^2 + \frac{1}{2}m_V v_V^2 - m_P\frac{\mu_S}{|\bar{r}_{PS}|} - m_V\frac{\mu_S}{|\bar{r}_{VS}|} - 0 \qquad \text{(E7)}$$

In this form, the energy takes the form of 2 Kepler problems : The vehicle and the planet about a fixed Sun. As the vehicle and the planet are roughly at a the same distance from the Sun just before and after the flyby, the terms $\frac{\mu_S}{|\bar{r}_{PS}|}$ and $\frac{\mu_S}{|\bar{r}_{PS}|}$ are almost equal and combine to : $\frac{\mu_S(m_P + m_V)}{|\bar{r}_{PS}|}$. Only with this approximation can the mechanism of a gravity assist be explained by an exchange of kinetic energy between the planet and the satellite.

Review Hyperbola

In this section we summarize the formulae for hyperbolic motion. The table below compares them to the better known results for elliptic motion. The Cartesian equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

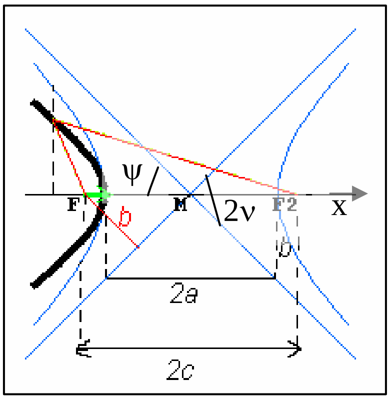


Fig.2- Hyperbola

gives both branches of the hyperbola. The x-axis pointing to the right, carries the two foci. To have the x-axis pointing to the periapsis, from the center of attraction (F_1), we consider only the left branch. In this way,the eccentricity or Laplace vector has its usual meaning. The corresponding parametric equation is :

$$x = -a\, ch\, H \quad \text{where ch=cosh; sh=sinh} \qquad \text{(H0)}$$
$$y = b\, sh\, H$$

We take a,b,c,p as line segments (>0), when they appear in coordinates, the sign is added when needed.

So, we write also : $E = \frac{\mu}{2a}$.

The polar equation from the attracting focus is , of course: $r = \frac{p}{1 + e\cos\theta}$

with e = c / a > 1. For the direction cos θ^* = -1/e , r becomes infinite.

	Hyperbola **$r_2 - r = 2a$**	Ellipse **$r + r_2 = 2a$**
Parameters	$c^2 = a^2 + b^2;\, p = \frac{b^2}{a} = a(e^2 - 1)$	$a^2 = c^2 + b^2;\, p = \frac{b^2}{a} = a(1 - e^2)$
p	$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{r_p} - \frac{1}{r_2}\right)$	$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{r_p} + \frac{1}{r_a}\right)$
r_p	$c - a = a(e-1)$	$a - c = a(1-e)$
Coordinates second focus	{ 2c, 0}	{-2c, 0}
r (E,H) from focus **r_2** from empty focus	$r = c\ ch\ H - a$ $r_2 = c\ ch\ H + a$	$r = a - c\cos E$ $r_2 = a + c\cos E$
Range θ	$\{-\theta^*, \theta^*\}$ $\cos\theta^* = -1/e$	$\{0, 2\pi\}$
{ x, y}	$x = a\,(e - ch\ H)$ $y = b\ sh\ H$	$x = a\,(\cos E - e)$ $y = b \sin E$
Cos θ	$\frac{e - ch\ H}{e\ chH - 1}$	$\frac{\cos E - e}{1 - e\cos E}$
Sin θ	$\sqrt{e^2 - 1}\ \frac{sh\ E}{e\ ch\ H - 1}$	$\sqrt{1 - e^2}\ \frac{\sin E}{1 - e\cos E}$

Table 1 - Hyperbola and Ellipse

When a particle runs through the hyperbola, its velocity turns over an angle 2ν with $\nu = 90-\psi$.
Note that the perpendicular from the focus on an asymptote equals the parameter b.
The formulae containing only {r, θ} are valid for both the elliptic and the parabolic case.

For the averaging procedure we will use the velocity components

$$\begin{aligned} v_r &= \dot r = \sqrt{\mu / p}\ e\sin\theta\,;\, v_n = \sqrt{\mu p}\ / r = \sqrt{\mu / p}\ (1 + e\cos\theta) \\ v_x &= -\sqrt{\mu / p}\ e\sin\theta; v_y = \sqrt{\mu / p}\ (e + \cos\theta) \\ v &= \sqrt{\mu / a}\ \sqrt{r_2 / r} \end{aligned} \tag{H1}$$

The time dependency is given the adapted Kepler equation :

$$\begin{aligned} M &= nt = e\sinh H - H \qquad n = \sqrt{\frac{\mu}{a^3}} \\ ndt &= (e\cosh H - 1)dH = \frac{r}{a}dH \end{aligned} \tag{H2}$$

The energy equation defines V∞ when r = ∞ as : $E = \frac{1}{2}V_\infty^2 = \frac{\mu}{2a}$ (H3)

Sphere of Influence

Where is the spacecraft (subscript V) close enough to the Earth that we may neglect the attraction of the Sun ? The attraction of the Sun on the moon is much stronger as the attraction from the Earth and nevertheless the moon is in the domain of attraction of the Earth. So, a direct comparison of the strength of the attraction of Sun and Earth does not give the correct answer. We have to compare the attraction of the Earth with the difference in attraction on the Earth and the Moon by the Sun. For a flyby we write the exact equations of motion of the vehicle twice. Once w.r.t. to the planet and a second time w.r.t. the Sun. Neither the planet or the Sun are an inertial point. So these equations are obtained by subtraction of the EOM's w.r.t. an inertial point

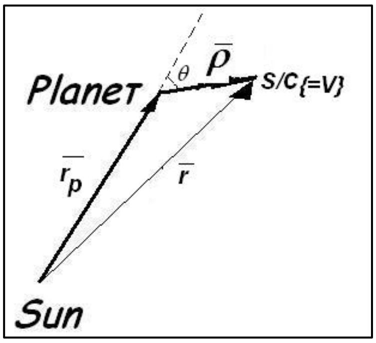


Fig.3 - Sphere of Influence .

In formulae with the notations of fig.3 above :

w.r.t. planet : $$\ddot{\bar{\rho}} + \bar{a}_P = \bar{d}_S \qquad \bar{a}_P = \frac{\mu_E}{\rho^3}\bar{\rho} \qquad \bar{d}_S = \mu_S\left(\frac{\bar{r}_p}{r_p^3} - \frac{\bar{r}}{r^3}\right) \qquad \text{(S1)}$$

w.r.t Sun $$\ddot{\bar{r}} + \bar{a}_S = \bar{d}_P \qquad \bar{a}_S = \frac{\mu_S}{r^3}\bar{r} \qquad \bar{d}_P = -\mu_P\left(\frac{\bar{\rho}}{\rho^3} + \frac{\bar{r}_p}{r_p^3}\right) \qquad \text{(S2)}$$

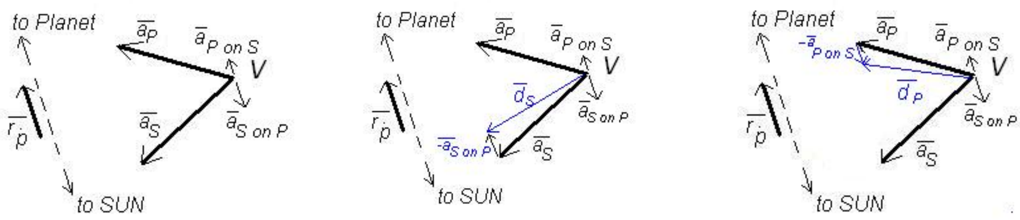


Fig.4a,b,c - Motion of the Vehicle w.r.t. Planet and Sun.

Both EOM's (S1) and (S2) are exact. For the motion of the vehicle w.r.t. the planet (Fig. 4b) we have, of course, (**a_p**) the attraction of the planet but also the term (**d_S**) due to the Sun. This term equals the difference in attraction on planet and vehicle by the Sun**.** Fig.4c shows the same situation for the motion w.r.t. the Sun : (**a_S**) is the attraction by the Sun and (**d_P**.) is the difference in attraction on Sun and vehicle by the planet . To determine the domain of influence we have to compare the ratio's :

$$\boxed{\frac{|\bar{d}_p|}{|\bar{a}_s|} = \frac{|\bar{d}_S|}{|\bar{a}_p|}}$$

Using (S1) in (S2), we can derive an equation $\rho\ (\theta)$ that defines the domain of attraction in terms of the mass ratio Sun/planet and r_P, the distance planet-Sun. For the substitution we use the eqn.'s (S3)

$$\bar{r}_p \cdot \bar{\rho} = r_p \rho \cos\theta = r_p^2\ u \cos\theta \qquad \text{(S3)}$$

$$\bar{r}_p \cdot \dot{\bar{r}} = r_p^2 (1 + u\cos\theta) \qquad u = \frac{\rho}{r_p}$$

to eliminate **r** while $$r^2 = r_p^2(1 + 2u\cos\theta + u^2)$$ exploiting the fact that $\rho \ll r_P, r$. Each ratio in (S2) is then :

$$f1(u,\theta) = \frac{|\bar{d}_p|}{|\bar{a}_S|} = \frac{m_P}{m_S}\left(1 + \frac{2\cos\theta}{u} + \frac{1}{u^2}\right)\sqrt{1 - 2\cos\theta\, u^2 + u^4} \cong \frac{m_P}{m_S}\frac{1}{u^2} \qquad \text{(S4)}$$

$$f2(u,\theta) = \frac{|\bar{d}_S|}{|\bar{a}_P|} \cong \frac{m_S}{m_P} u^3 \sqrt{1 + 3\cos^2\theta}$$

Fig.5 shows the ratio's f1, f2 for the Sun-Earth case, m_S / m_E = 332 946. Left of the intersection point { u ~.0055, f1=f2=.1044} , the influence of the planet over the attraction by the Sun (f1) is much stronger as the influence of the Sun over the attraction of the planet (f2) . So the motion of the vehicle is strongly dominated by the planet. We are in the sphere of influence of the planet.

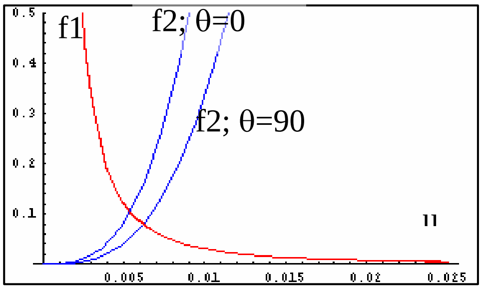


Fig.5 - Ratio's influence/ attraction

The intersection point follows from :

$$(\rho) = R_{SOI} = r_P \left(\frac{m_P^2}{m_S^2 \sqrt{1+3\cos^2\theta}} \right)^{1/5} \approx r_{SP} \left(\frac{m_p}{m_S} \right)^{2/5} \qquad \text{(S5)}$$

The surface $R_{SOI}(q)$ defined by (S5) is a surface of revolution about the Sun-Planet direction and is shown in Fig.(6). In the direction of the Sun it is $1/\sqrt[10]{4} = .87$ of the value perpendicular to the Sun.

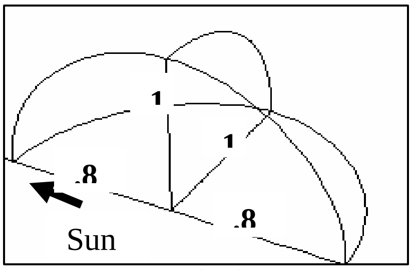


Fig.6 - Domain of influence

The standard practice is to replace this surface by a sphere. For the Earth, the 2 values for θ = 0, 90 are respectively :

824 951.8 km and 924 646.8 km.

Standard Patched Conics procedure

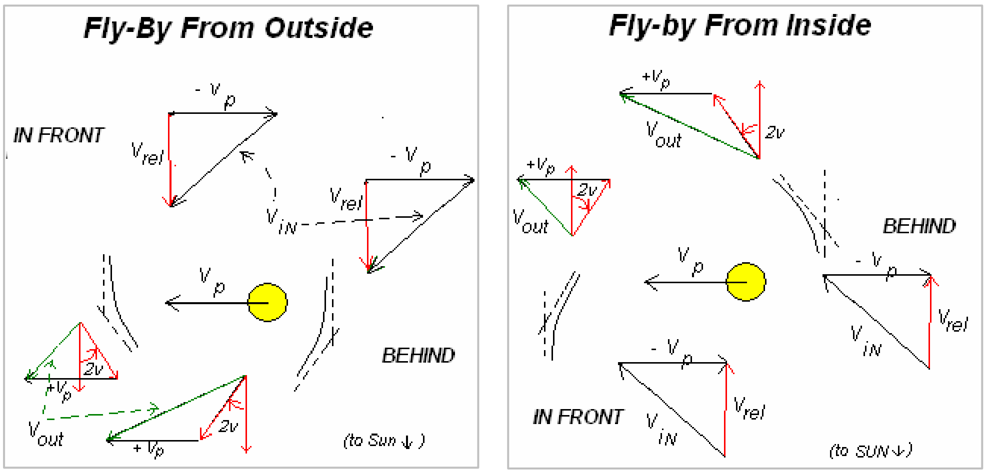


Fig 7a,b - Flyby Summary - (Orbit normal out of the paper - Planet rotates anti-clockwise)

After this digression on the energy and the radius of the sphere of influence we return in figures 7 to the patched conics procedure described in fig.1. The planet has velocity V_p. In phase I, the incoming velocity of the vehicle, or its velocity w.r.t. the Sun is V_i. When entering the sphere of influence, the relative velocity to the planet $V_r = V_i - V_P$. Inside the sphere of influence, the vehicle follows a perfect hyperbola as we consider only the attraction of the planet. So, the value V_r is taken as V_∞ . This is standard meaning of V_∞ although strictly speaking V_∞ should be computed from :

$$\frac{1}{2}V_\infty^2 = \frac{1}{2}V_R^2 - \frac{\mu_{Planet}}{R_{SOI}} \qquad \text{(PC0)}$$

Anyway, V_∞ gives already one parameter of the hyperbola, by eqn.(H3) namely the semi-major axis a. As mentioned above, the second parameter b is the miss-distance from V_r to the center of the planet. With a and b known we have :

$$c = \sqrt{a^2 + b^2};\ e = c/a;\ \cos\psi = \sin\nu = 1/e\ ;$$
$$r_p = c - a \quad h_p = r_P - R_E > 0 \qquad \text{(PC1)}$$

the turning angle 2ν of V_∞ and the flyby altitude that must be positive to avoid crashing on the planet. This condition is easily rewritten as :

$$\boxed{\sin\nu \le \frac{1}{1 + \dfrac{v_\infty^2}{v_{grazing}^2}} \quad v_{grazing} = \sqrt{\frac{\mu}{R_{planet}}}} \qquad \text{(PC2)}$$

The maximum turning angle is the parabolic approach when $V_\infty = 0$, sin ν =1 , 2 ν =180.

$$\text{When} \quad v_\infty = \frac{v_{grazing}}{2} \Rightarrow \nu = 53.13\,\deg \quad \Rightarrow \quad 2\nu = 106.26 \quad \deg$$

The rotation of V_∞ is always towards the planet that attracts the spacecraft. In both figures, the Sun is at the bottom of the page so in the left figure 6a, the spacecraft is initially further away from the Sun. If , without attraction from the planet, it would pass behind the planet, the relative velocity turns clockwise due to the attraction of the planet. If it would pass before the planet, V_∞ rotates anti-clockwise. For the transition to phase III, we add again V_P to the rotated V_∞. The figure 6a shows that after passing behind the planet the obtained inertial velocity has increased. After a passage before the planet, the inertial velocity has decreased. The figure 6b shows that the effect of the flyby remains the same when the planet is approached from the inside. The outgoing velocity is maximised when V_∞ is turned to align with the velocity of the planet. This is not necessarily the maximal turning angle.

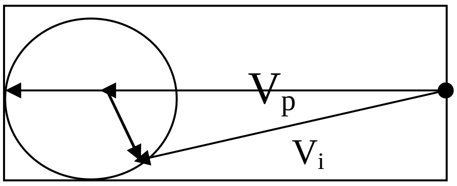


Fig.8 - Maximal Increase Velocity

After adding V_P to the rotated V_∞, the state vector after the flyby {$r_{inertial} \sim r_{planet}$ }, defines the new orbit (phase III) .

The standard patched conics technique is an excellent tool to design trajectories in the solar system. However a real flyby shows small deviations from this idealized picture . In the sequel we have a closer look at these small deviations.

Time of Flight in Sphere of Influence

The duration of phase II is easily calculated starting from eqn's (H3 , PC1) :

$$\boxed{\cosh H = \left(\frac{r_{SOI}}{|a|} + 1\right)\frac{1}{e} \quad \Rightarrow H \quad \Rightarrow \quad T = \frac{a}{V_\infty}\left(e\sinh H - H\right)} \qquad \text{(PC5)}$$

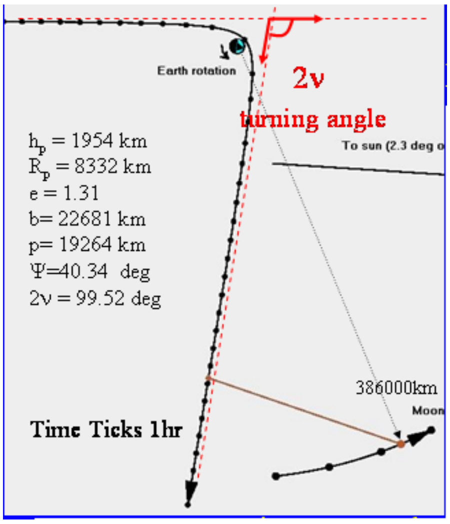
where T is the time from R_{SOI} to the perigee passage. For the data of the Rosetta flyby, given below we have :

a = 26704.055 km ⇔ V_inf = 3.863492 km/s e = 1.312005 ; r_{SOI}= 924646.8 km

⟹ H = 3.994318941 ⟹ $T_{To\ Perigee}$= 218465.670 sec = 2.528538 days

and the total time spent in the sphere of influence is about 5 days.

Data of Rosetta flyby on the Earth 4 Mar 2005



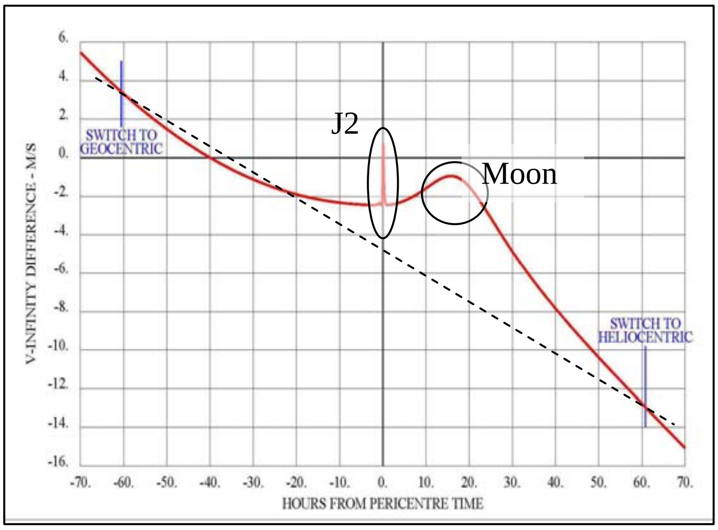


Fig.9a,b - Rosetta flyby and data

Figures 9a is taken from **[T.Morley, F.Budnik ,2006]** and shows phase II of the flyby of the Rosetta spacecraft on 4 Mar 22h 2005. T. Morley placed the reconstituted orbit and fig 9b at our disposal. The data file is a list of complete state vectors in the EME2000 frame with a variable time step in TDB. Outside the SOI, the data are Sun-centered, inside the SOI they are Earth-centered. Figure 9b shows the evolution of the deviations of the instantaneous V_∞ from its osculating value at the perigee passage. The most striking global feature is the decrease in average of V_∞ during the passage through the sphere of influence (dotted line on fig.9b). In the standard patched-conics procedure, this value is constant which implies that the incoming and outgoing asymptote belong to the same hyperbola. In the real flyby trajectory V∞ or the "energy of the spacecraft w.r.t. the Earth " has changed. [**Anderson, Campbell]** consider this is as a surprise .There are also two local effects : a peak 1 hour before to 1 hour after the perigee passage (J2) and a bump 10-20 hours after the perigee passage (Moon).

Event	Time (TDB)	Sec from perigee	R (km)	V_∞ (km/s)	ΔV_∞ m/s
Entrance SOI	02 Mar 09:36:59.980	-217 998.320	922 722.194	3.866 908	3.414
Perigee passage	04 Mar 22:10:18.300	0	8331.798	3.863 494	0
Leaving SOI	07 Mar 11:00:33.831	+219 015.531	925 692.344	3.850 546	-12.948

Table 2 - SOI Data

Table 2 shows an average decrease of 16 m/s of V_∞ from the entrance in the SOI to the exit.

The average decrease of V_∞ is .1347856 [m/s] / h or 3.234855 [m/s] /day

Event	a (km)	b	p	e	h_p
Entrance SOI	26656.9	21816.2	17854.6	1.2922	1411.12
Perigee passage	26704.1	22680.3	19262.8	1.312	1953.66
Leaving SOI	26884.4	22721.7	19203.8	1.309	1937.64

Table 3 - Osculating elements from reconstructed orbit after flyby.

The data show that the flyby altitude extrapolated from the osculating hyperbola, determined from the state vector at the entrance of the SOI, is about 540 km lower as the flyby altitude of the real trajectory given in Table III. A realistic prediction of the flyby altitude can only be based a more precise model for the trajectory in the SOI. The same holds for the prediction of the time of perigee passage. Table 4 shows the time to perigee from the osculating hyperbola at the entrance deviates 284" from the observed perigee passage. The error on the predicted time of the perigee passage is reduced to 17" when the osculating parameters at perigee are used. During operations, the predicted trajectory is based on a model including the harmonics of the Earth and the attraction from the Sun and Moon. With such model, the error on the time to perigee , predicted 2 weeks in advance, was only .34 sec **[T.Morley, F.Budnik ,2006]**.

Parameters {e,a} at	$H_{\text{Kepler eq.}}$	T(sec)	T_{observ}	ΔT
Entrance SOI	4.00923	217714	217998	284 early
Perigee passage	3.99229	217981	"	17 early

Table 4. - Time from entrance SOI to perigee

Effect of J2

One can expect that the peak in V_∞ around perigee is due to the effect of J2 on the trajectory.

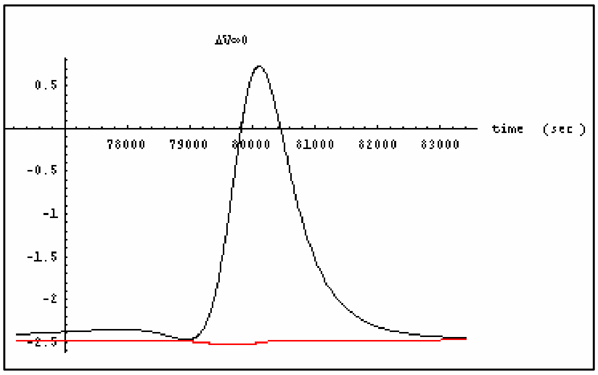


Fig.10 a -$\Delta V_{\text{infinity}}$ (m/s) below 26000 km

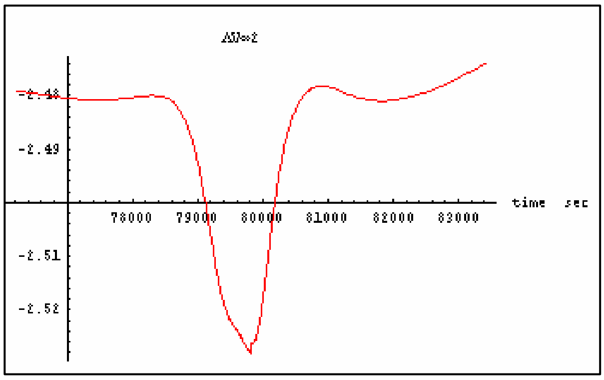


Fig.10 b - $\Delta V_{\text{infinity}}$ (m/s)below 26000 km - zoom

Therefore we look in fig.10a at a zoom of fig.9b when the distance to the center of the Earth is below 26000 km which is about 50 minutes before and after the perigee passage. The amplitude of the peak in V_∞ is about 2.5 m/s. When we include the potential term coming from J2, and compute V_∞ from the corresponding energy equation (D1) , the peak disappears almost completely. V_∞ remains almost constant at a value -2.5 m/s below the reference value and the remaining total variation 4cm/s.

$$V_\infty^2 = V^2 - \frac{\mu}{r} - \frac{f^2}{3}\frac{\mu}{r^3}\left(1 - 3\sin^2 decl\right) \quad f^2 = 3J_2 R_E^2 \quad \cos decl = z/r \qquad \text{(D1)}$$

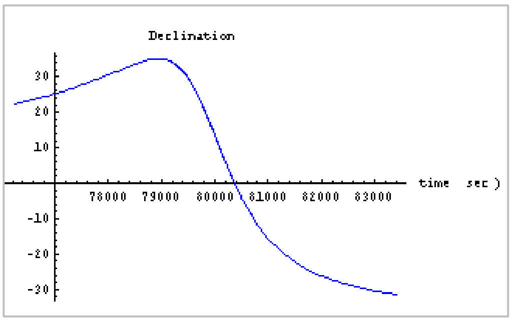


Fig.11 a -Declination (deg) below 26000 km

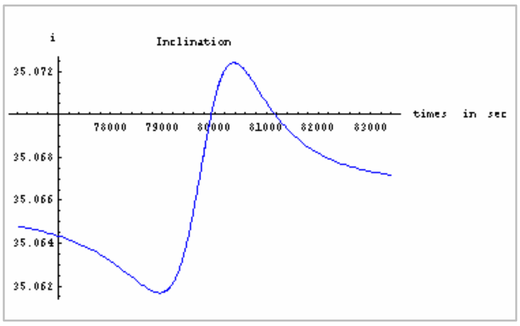


Fig.11 b -Inclination below 26000 km

As mentioned in **[T.Morley, F.Budnik ,2006]** an fictitious perigee manoeuvre of .76 mm/s, giving a ΔV_∞ of 2mm/s was inserted to improve the orbit reconstitution.
We just mention here that the exact trajectory for zero declination, when J2 is included, is an hyperbola modified by a constant value in the radial direction. It is a fixed curve of the 4th degree that precesses slowly[**Janssens,2003**]. When the declination increases(for Rosetta the declination is about 23 deg as the trajectory is almost in the ecliptic), the precession slows down and a tiny inclination change comes in.
The effect of J2 plays only during 33 minutes or 16 minutes before and after the perigee passage or, more relevant, when the altitude is below 4100 km. J2 explains the peak in Fig.9b almost completely but the change in the slope of V_∞ starts 20hrs before perigee.

Perturbation by the Sun

It is expected that the slope in V_∞ is caused by the perturbation of the Sun. The attraction of the Sun can not be switched off as is done in the patched conics model. One way to investigate the perturbations by the Sun is to replace the Kepler problem of phase II by a restricted circular 3BP. In other words, instead of considering only the attraction of the planet we investigate the trajectory of the vehicle under the influence of the planet and the Sun when the planet is moving on a fixed circular orbit about the Sun. Another approach is to apply the perturbation technique to the hyperbolic orbit inside the sphere of influence. The standard application of the perturbation technique is indeed for elliptic orbits where the perturbation is averaged over a period [**Cornelisse, Janssens,1983**]. As an hyperbolic orbit is not periodic, we average the perturbation over a range symmetric around perigee : [-H, H] . For the flyby it is logical to take the H value that corresponds to the SOI. In some special cases, a finite result is obtained when H goes to infinity. A short review of the averaging technique is included.

Restricted Circular 3BP

The restricted circular 3BP is , since Poincaré , probably the most investigated problem in orbital dynamics. In this model the motions of the 2 primaries (Sun and planet) are the exact solution of a 2BP and hence known time functions (prescribed) .The attraction from the vehicle on the primaries is (justifiable) neglected . This implies that we have introduced tiny constrained forces on the Sun and the planet that compensate the attraction of the vehicle. As a consequence, there is no conservation of energy . The EOM of the vehicle w.r.t. the primaries are still given by (S1) and (S2) where $\bar{r}_p$ is now a known function of time. When we take the planet as primary :

$$\ddot{\bar{\rho}} + \frac{\mu_P}{\rho^3}\bar{\rho} = \mu_S\left(\frac{\bar{r}_p(t)}{r_p^3(t)} - \frac{\bar{r}_p(t)+\bar{\rho}}{\left|\bar{r}_p(t)+\bar{\rho}\right|^3}\right) \qquad \text{(R1)}$$

This EOM can be derived from the following Lagrangian with a time dependent potential [**Broucke , 1990**] :

$$\begin{aligned} L &= T + U(t) = \frac{1}{2}\dot{\bar{\rho}}^2 + \frac{\mu_P}{\rho} + \mu_S\ \Re(\bar{\rho},t) \\ \Re(\bar{\rho},t) &= \frac{1}{\left|\bar{r}_P(t)+\bar{\rho}\right|} + \frac{\bar{r}_P(t).\bar{\rho}}{r_P^3(t)} \end{aligned} \qquad \text{(R2)}$$

The energy is : $E = T - U \quad \text{and} \quad \dot{E} = \frac{\partial E}{\partial t} = -\mu_S \frac{\partial \Re}{\partial t}$

When the planet is in a circular orbit, with normal $\bar{1}_n$ and angular velocity Ω, $|\bar{r}_p| = r_P$ is constant and it is easily shown that

$$\frac{\partial \Re}{\partial t} = \Omega_P\left[\frac{r_P}{r(t)^3} - \frac{1}{r_P^2}\right]\left(\bar{\rho}\times\bar{r}_P\right)\bar{1}_n \qquad \text{(R3)}$$

On the other hand, one shows that the angular momentum $\bar{h} = \bar{\rho}\times\dot{\bar{\rho}}$ has up to a factor Ω the same rate of change. Hence the quantity

$$\boxed{J = E - \Omega\, h_n} \qquad \text{(R4)}$$

known as the Jacobian integral is conserved during the motion. A frame rotating such that the Sun-Planet direction is a reference axis is not needed in this derivation. The Jacobian integral shows that there is a fixed relation between the change in energy and the change of the angular momentum component perpendicular to the orbital plane of the planet [**Anderson, Campbell]**.

$$(E_1 - E_0) = \Omega\left(h_{z1} - h_{z0}\right) \quad \Omega = \text{angular rate Planet} \qquad \text{(R5)}$$

[Tisserand,1889] wrote the Jacobian integral as a relation between the elements { a,p, i } before and after the flyby :

$$\boxed{\frac{1}{a_0} + \frac{2}{r_P}\sqrt{\frac{p_0}{r_P}}\cos i_0 = \frac{1}{a_1} + \frac{2}{r_P}\sqrt{\frac{p_1}{r_P}}\cos i_1} \qquad \text{(R6)}$$

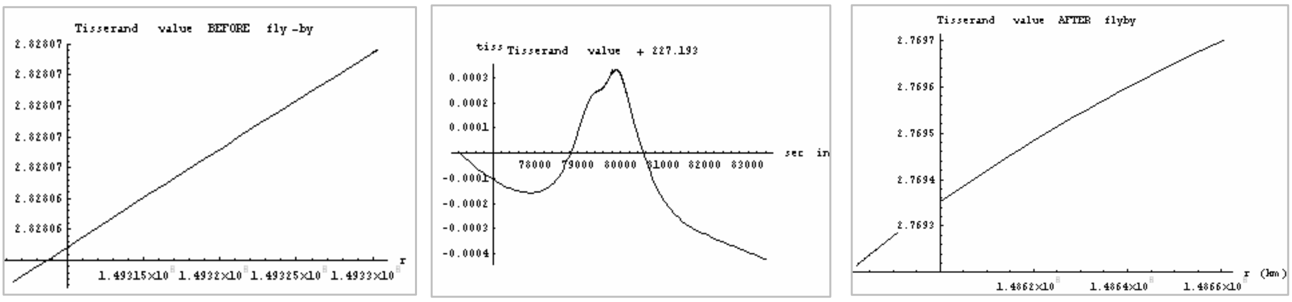


Fig12. - Evolution Tisserand/Jacobian Constant during flyby

When the planet is in a circular orbit, the energy can also be written as :

$$E = \frac{1}{2}\dot{\bar{\rho}}^2 - \frac{\mu_P}{\rho} - \mu_S\left(\frac{1}{|\bar{r}_P(t) + \bar{\rho}|} + \frac{\bar{r}_P(t).\bar{\rho}}{r_P^3(t)}\right)$$

$$= E_H - \frac{\mu_S}{r(t)} - \Omega_P^2\left(\bar{r}_P(t).\bar{\rho}\right) = \frac{1}{2}V_\infty^2 - \frac{\mu_S}{r(t)} - \Omega_P^2\left(\bar{r}_P(t).\bar{\rho}\right)$$

and $\Delta E = \frac{1}{2}V_{\infty-out}^2 - \frac{1}{2}V_{\infty-in}^2 - \frac{\mu_S}{r_{out}} + \frac{\mu_S}{r_{in}} - \Omega_P^2\bar{r}_P\left(\bar{\rho}_{out} - \bar{\rho}_{in}\right) - \Omega_P\left(h_{z-out} - h_{z-in}\right)$ (R7)

and in first approximation ($r_{out} = r_{in}$, $\bar{r}_p$ =constant) and 2 symmetric points on a reference hyperbola :

$$\Delta V_\infty V_{\infty-in} = 2\,\Omega_P^2\; r_P b \sinh H_{SOI}\,\cos\left\langle r_P, y_{axis-hyperbola}\right\rangle \qquad \text{(R8)}$$

when the sun-planet direction is aligned with an asymptote of the hyperbola $\cos\left\langle r_P, y_{axis-hyperbola}\right\rangle = -b/c$ and we have :

$$\Delta V_\infty = -2\frac{\Omega_P^2\; r_P}{V_{\infty-in}}\frac{p}{e}\sinh H_{SOI} \qquad \text{(R9)}$$

This approximation is too crude for numerical applications but the structure is interesting.

Averaging

For any quantity x, the average or mean value over a period T is :

$$\hat{x} = av[x] = \frac{1}{T}\int_0^T x(t)\; dt \qquad \text{(A1)}$$

An illustration for elliptic motion is the mean distance over the period P = 2π / n :

$$\hat{r} = \frac{1}{P}\int_0^P r(t)\; dt = \frac{n}{2\pi}\int_0^{2\pi} r(E)\frac{(1 - e\cos E)}{n}dE = a\left(1 + \frac{e^2}{2}\right) \qquad \text{(A2)}$$

The same procedure on (1 / r) gives $av[r^{-1}] = \frac{1}{a}$. The average of an inverse is not the inverse of the average. Also the average with respect to another variable , for instance, dθ, gives a different result. A list of average values in elliptic motion is given by **[Tisserand 1889]**. The table below gives the corresponding results for hyperbolic motion.

As we are mainly interested in time averages, the basic tool to calculate the integrals is the substitution dt by dE or dH as given by eqn (H2).

average	Ellips Over a period	Hyperbola **For a passage {H,-H}**
$\hat{x} = \mathrm{av}[r\cos\theta]$	-3/2 a e	$a\dfrac{3eH - 2(1+e^2)shH + e\,shHchH}{2(eshH - H)}$ $\sim a/2\;chH$ H=>∞
$\hat{y} = \mathrm{av}[r\sin\theta]$	0	0 for any H
$\mathrm{av}\,[\sin\theta]$	0	0 for any H
$\mathrm{av}\,[\cos\theta]$	-e	$-\dfrac{eH - shH}{e\,shH - H}$ ~ 1/e H => ∞
$\mathrm{av}\,[r\cos^2\theta]$	a(1/2+e^2)	$a\dfrac{-(1+2e^2)H + 4eshH - shHchH}{2(eshH - H)}$ $\sim -a/(2e)\;chH$ H =>∞
$\mathrm{av}\,[r\sin^2\theta]$	a(1/2-e^2)	$p\dfrac{H - shHchH}{2(eshH - H)}$ $\sim -p/(2e)chH$ H =>∞
$\mathrm{av}\,[r\sin\theta\cos\theta]$	0	0 for any H

Table 5. - averaged quantities

Let the perturbing acceleration be denoted a_{cc} => [a_{cc} cos α , a_{cc} sin α]
where α is angle with the axis of the hyperbola. The dot product of the perturbation with the velocity gives the local change of energy :

$$\boxed{dE = -\frac{\mu}{2a^2}da = \bar{a}_{cc}\cdot\bar{v} = a_{cc}\sqrt{\frac{\mu}{p}}\big(-\cos\alpha\sin\theta + \sin\alpha(e+\cos\theta)\big)} \qquad \text{(A3)}$$

In (A3), the values of $\mathbf{a_{cc}}$ and α can be fixed or time dependent, depending on the nature of the perturbation. For illustration we treat first the case of a constant perturbation.
The averaged rate of change of a is :

$$d\hat{a} = -\frac{a_{cc}\,a^2}{T}\sqrt{\frac{1}{\mu\,p}}\int_{-T}^{+T}\big(-\cos\alpha\sin\theta + \sin\alpha(e+\cos\theta)\big)dt \qquad \text{(A4)}$$

Using the results for av[sinθ] and av[cosθ] from the table 5 above, the averaged rate of change of a over a period in elliptic motion is zero. For a complete passage of the hyperbolic motion (infinite time), the result is:

$$d\hat{a} = -\frac{a_{cc}\sin\alpha}{\sqrt{\mu/p}}\,\frac{a}{e} \qquad \text{(A5)}$$

So, when the perturbing acceleration is along the major axis (α = 0), the averaged rate of change is zero , so a and the energy remain unchanged as expected by symmetry. When the constant perturbation has a component on the minor axis, the energy will increase or decrease according to the sign of α.
The perturbation of the Sun during the flyby is given by eqn. (S1) and in first order in ρ :

$$\bar{a}_{cc} = d_S = \mu_S\left(\frac{\bar{r}}{|\bar{r}|^3} - \frac{\bar{r}_P}{|\bar{r}_P|^3}\right) \cong -ns^2\bar{\rho} + 3ns^2(\bar{\rho}.\bar{1}_P)\,\bar{1}_P \qquad ns^2 = \frac{\mu_S}{r_P^3} = \Omega_P^2 \qquad \text{(A5)}$$

where ns^2 is the square of the angular velocity of the Earth (~1 deg/day) and only the first term of the development :

$$\frac{1}{r^3}=\frac{1}{r_P^3}-3\frac{\bar{\rho}.\bar{1}_P}{r_P^4}-\frac{3}{2}\frac{\rho^2}{r_P^5}+15\frac{(\bar{\rho}.\bar{1}_P)^2}{r_P^5}.$$

is used.

For the example we treat $\bar{1}_P$, the unit vector from the Sun to the Earth as a constant direction making an angle α with the x-axis of the hyperbola.

In the axis system of the hyperbola (fig.2), we have : $\bar{1}_P=\cos\alpha\,\bar{1}_x+\sin\alpha\,\bar{1}_y$ (A6)

Let b be the angle β between the radius vector ρ and $\bar{1}_P$ then : $\cos\beta=\cos(\alpha-\theta)$

So, (A3) takes the form :

$$\frac{dE}{dt}=V_\infty\frac{dV_\infty}{dt}=\bar{a}_{cc}.\bar{v}=-\rho\,ns^2(\cos\theta\,v_x+\sin\theta\,v_y)+3ns^2\rho\cos\beta(\cos\alpha\,v_x+\sin\alpha\,v_y) \tag{A7}$$

where $v_x=-\sqrt{\mu/p}\;e\sin\theta;\quad v_y=+\sqrt{\mu/p}\;(e+\cos\theta)$

The hyperbola is described clockwise so the real velocity components have the opposite sign as in eqn (H1) .
Substituting the velocity components in the first term of (A7) that originated from a radial perturbation we have :

$$-ns^2\sqrt{\frac{\mu}{p}}[-e\,\rho\sin\theta\cos\theta-\rho\sin\theta\cos\theta-e\rho\sin\theta] \tag{A8}$$

All the terms in (A8) are odd terms and average out to zero. This result is in agreement with the fact that an additional radial term can not change the angular momentum and from the Jacobian integral follows that no change in angular momentum implies no change in energy when the hyperbola is in the orbital plane of the planet.
The second term of (A7) is the perturbation that is aligned with the Sun-Planet direction. Substitution of the velocity components gives , after disregarding again the odd terms :

$$V_\infty\frac{dV_\infty}{dt}=-3ns^2e\frac{\sin 2\alpha}{2}\sqrt{\frac{\mu}{p}}\left[\rho\cos\theta-\rho\sin^2\theta+\frac{\rho\cos^2\theta}{e}\right] \tag{A9}$$

The slope of dV∞ is the averaged value :

$$\mathrm{av}<\frac{dV_\infty}{dt}>=3\,ns^2\,\frac{\sin 2\alpha}{2}\;\frac{\mathrm{c}}{\mathrm{b}}\,\mathrm{av}\left\langle\rho\cos\theta+\rho\frac{\cos^2\theta}{e}-\rho\sin^2\theta\right\rangle \tag{A10}$$

where the sign has been inverted again as the hyperbola is described in the sense of decreasing anomaly and decreasing H. The averaged values in the right hand side can be taken from Table 4. Substituting the values for H large we obtain :

$$\mathrm{av}<\frac{dV_\infty}{dt}>=\frac{3}{4}ns^2\sin 2\alpha\;\;\mathrm{a}\frac{\mathrm{c}}{\mathrm{b}}\left\langle 1+\frac{e^2-1}{e}-\frac{1}{e^2}\right\rangle\cosh H=\frac{3}{4}ns^2\sin 2\alpha\;\;\mathrm{b}\frac{(e+1)}{e}\cosh H \tag{A11}$$

when r_p is aligned with the incoming asymptote ($\cos\alpha= -1/e$) and $(\sin 2\alpha)/2 = -ab/c$
For H=4 and the hyperbolic parameters as in Fig.9a formula (A10) gives

DV∞ / day = - 2.983 m/s or -14.915 in 5 days.

which , taking all the assumptions into account, is in good agreement with the observed decrease of V_∞ during the flyby. The approximation for large H given in (A11) gives -2.767 m/s per day or 13.8 m/s in 5 days.

Remark

The paper by **[Morley, Budnik]** showed that a tiny impulsive manoever at perigee of .67 mm/s was needed the fit the incoming and outgoing trajectory exactly. Before interpreting this result as showing a defect in Newton's gravitational law , one must remember that working with ephemerides for the planets is strictly speaking equivalent to using a restricted 3 (or N) -body problem. In the latter case, the quantity conserved is the Jacobian integral while in the general 3BP the energy is conserved. It remains to be seen that this anomaly can or can not be explained by the distinction between the two conservation laws in these two models.

Conclusion

In the standard patched conics procedure only the attraction of the planet is considered when the spacecraft is in the sphere of influence of the planet. This procedure is very useful for the design of trajectories in the solar system. As a consequence the incoming and outgoing V_∞ are strictly equal. In a more precise analysis, the trajectory is still perturbed by the Sun. A formula for estimating the magnitude of this effect was derived in this paper. The agreement with the observed change in V_∞ during the Rosetta flyby is good. It is also shown that the effect of J2 has a much shorter duration and has a tendency to cancel out.

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