Analysis of Equilibrium Points of the Three Body Problem for the Formation of Rotational of Dumbbell-shaped Body

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Abstract

This paper describes the stability of the Rotational Dumbbell shaped body. This problem, defined as the Rod-Connected Restricted Three Body Problem, is the special case of the Restricted Three-Body Problem, characterized by three bodies: two primary bodies connected each other by a rigid stick; and a mass-free body. While the Restricted Three-Body problem has only one degree of freedom, there are two degrees of freedom in this problem. By setting properly two parameters, described as the non-dimensional angular velocity and the mass ratio, both positions of equilibriums and stable regions are determined. In this paper, the stability in any type of those non-dimensional values is analyzed. There are five equilibriums and three stable points. Of these equilibriums, three points are along the line passing through two primary bodies, and the other two points are along the perpendicular bisector between two primary bodies. In addition, those stable points are comprised of two points which are along the perpendicular bisector between two primary bodies and a point which is along the line segment between two primary bodies.

Keyword: Itokawa, rubble-pile, and three-body problem

制限三体問題を用いた回転するダンベル形状天体の安定性に関する解析

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概要

本稿では，ダンベル形状の物体まわりの，質量の無視できる質点（以降，第三質点と呼ぶ）の運動の安定性について考察する．ダンベル形状物体は，二つの有限質量質点が剛棒で接合されている物体（以降，二質点と呼ぶ）であると模擬できる．この問題は，制限三体問題の特別な問題であると考えることができる．二つの自由度が存在するため，本稿では，この二つの自由度を無次元角速度，無次元質量で表現した．これによれば，二質点を通る直線上に 3 つ，またこの二質点の垂直二等分線上に 2 つで，計 5 つの平衡点が存在する．また，これらの平衡点のうち，垂直二等分線上の二質点にはさまれた点において安定な領域があることが示される．

1 Nomenclature

μ_1, μ_2	= gravitational constant of two primary bodies
d_{i3}	= distance between the primary bodies
ω	= common angular velocity
n	= mean motion
(x, y)	= position of mass-free body
i	= $\sqrt{-1}$
U	= potential in rotating frame in the restricted three body problem
$\tilde{U}$	= potential in rotating frame in the rod-connected restricted three body problem
$(\delta x, \delta y)$	= perturbation around libration points

2 Introduction

In 2005, the "Hayabusa" spacecraft arrived at a near-Earth asteroid "Itokawa" and provided us its great important scientific results.

From this mission, it was found that Itokawa has been many peculiar characteristics: low bulk density, high porosity, rough surface with studded boulders, etc. Those characteristics indicate that Itokawa is a rubble pile asteroid. Also, a previous paper showed that Itokawa was considered to be a typical Near Earth Object [1]. It means that studying the formation of Itokawa is great important to uncover the planetary evolution scientifically. Thus, in this paper, by using celestial mechanics, this study focuses on how Itokawa has been shaped.

According to another previous paper, it was shown that Itokawa had been formed by the contact of two core asteroids which had belonged to a preexisting parent body [2]. Thus, the analysis is divided into two processes: the case (1) before and (2) after the contact of the two core asteroids. In the case before the contact, it is considered many large fragments to be re-aggregated by each gravitational force. Since those large fragments assumed to be mass particles, the motion is explained by the Multi-Body Problem. In the case after the contact, since the fragments are much smaller than the core asteroids, they can be assumed to be mass-free bodies. In this case, the core asteroids can be considered to be a dumbbell shaped body, modeled as two primary bodies which are connected with each other by a rigid stick.

The case after the contact can be discussed more easily than the case before the contact, because, in the case after the contact, it is not necessary to discuss the gravitational effects of each fragment. In this case, to specify the motion of fragments, it is necessary to discuss the stability of them. It is because whether this system has stable points or not causes to change the motion of fragments, and its trait is an important factor to form Itokawa shape. From these reasons, this paper focuses on the case after the contact, and describes the position of equilibriums and the stability of those positions around the dumbbell shaped body.

In this problem, in addition to gravitational forces from two primary bodies and centrifugal forces, the force from the stick affects the motion of two primary bodies. In other words, the force can fix the distance between two primary bodies regardless of any type of the common angular velocities. Therefore, this problem has two degrees of freedom. In this paper, the stability is specified in any type of two non-dimensional parameters: the non-dimensional angular velocity; and the mass ratio.

3 The restricted three-body problem

Before indicating the Rod-Connected Restricted Three-Body Problem, we introduce the normal Restricted Three Body Problem (RTBP). This theory makes it possible to discuss the stability in the Rod-Connected Restricted Three-Body Problem. First, the positions of equilibriums are discussed. Then, the stability of motions around the equilibriums is specified.

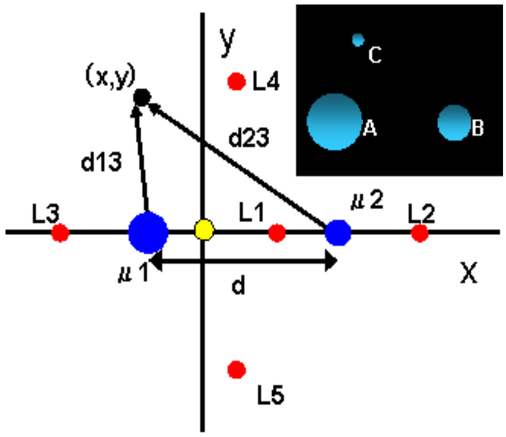

Figure 1: The libration points in the RTBP system; Three equilibriums exist along the x axis and other two equilibriums are at at the apex of an equilateral triangle.

3.1 Equation of motion

Let us consider a position of a mass-free body subject to the gravitational attraction from the primary bodies which are rotating with a constant angular velocity n, known as the mean motion. In this paper, two-dimensional surface is discussed because of its simplification of this discussion. The origin is the center of mass, x axis is defined as a line passing through both of the primary bodies, and y axis is perpendicular to x axis. This system, rotating with the primary bodies, is shown in Figure 1. Let us define U_x as,

$$U_x = \frac{\partial U}{\partial x} \tag{a}$$

In the rotating frame, the equation of motions are written by,

$$\ddot{x} - 2n\dot{y} = U_x \tag{1.1}$$
$$\ddot{y} + 2n\dot{x} = U_y \tag{1.2}$$

where,

$$U = \frac{\mu_1}{d_{13}} + \frac{\mu_2}{d_{23}} + \frac{1}{2}n^2\left(x^2 + y^2\right) \tag{2}$$

$$d_{13} = \sqrt{(x - x_1)^2 + y^2} \tag{3.1}$$
$$d_{23} = \sqrt{(x - x_2)^2 + y^2} \tag{3.2}$$

subject to

$$\mu_1 + \mu_2 = 1 \tag{4.1}$$
$$d = 1 \tag{4.2}$$
$$n = 1 \tag{4.3}$$

In these equations of motion, there is only one degree of freedom. Thus, once one variable is determined, the motion of the mass-free body can be described utterly.

3.2 Equilibriums

Because a mass-free body is not attracted by any forces at equilibriums, the condition at equilibruims is described as,

$$U_x = U_y = 0 \tag{5}$$

According to this condition, there are five equilibriums in this system. These points are known as the libration points and each point is defined as from L_1 to L_5. Three of them lie on the x axis, known as the collinear libration points. Other two points are at the apex of an equilateral triangle with a base formed by the line joining the primary bodies, known as the triangular libration points. These are shown in Figure.1.

Let us define two non-dimensional parameters as,

$$\chi = \frac{d_{23}}{d}, \quad \mu = \frac{\mu_2}{\mu_1} \tag{6}$$

Speaking generally, the position of the collinear libration points can be expressed only by using sum of series. However, by using these parameters, the relational expressions which are satisfied at the collinear libration points can be obtained.

The relational expressions of the collinear libration points are described as,

$$\frac{\chi^2}{(1-\chi)^2}\frac{(1-\chi)^3-1}{\chi^3-1} = \mu \tag{7.1}$$
$$\frac{\chi^2}{(1-\chi)^2}\frac{(1+\chi)^3-1}{\chi^3-1} = -\mu \tag{7.2}$$
$$\frac{\chi^2}{(\chi-1)^2}\frac{(\chi-1)^3-1}{\chi^3-1} = -\mu \tag{7.3}$$

Eq.(7.1) shows the position of L_1, Eq.(7.2) shows the position of L_2, and Eq.(7.3) shows the position of L_3.

On the other hand, the position (x_0, y_0) of L_4 and L_5 is written by,

$$(x_0, y_0) = \left(\frac{1-\mu}{2(1+\mu)}, \pm\frac{\sqrt{3}}{2}\right) \tag{8}$$

3.3 Stablity of the libration points

First, we difine U_{xx} as,

$$U_{xx} = \frac{\partial^2 U}{\partial x^2} \tag{b}$$

In the vicinity of the libration points, the linearized equations of motion is described as,

$$\delta\ddot{x} - 2n\delta\dot{y} = U_{xx0}\delta x + U_{xy0}\delta y \tag{9.1}$$
$$\delta\ddot{y} + 2n\delta\dot{x} = U_{xy0}\delta x + U_{yy0}\delta y \tag{9.2}$$

In these equations, Ths subscripts "0" means the respect to the libration points. Although in our system of units $n = 1$, we will retain n in the equations to emphasize that all the terms in the equations of motion are acceleration.

Suppose that $\delta x = Ae^{\lambda t}$, $\delta y = Be^{\lambda t}$, and $\lambda = \sigma + i\psi$ in Eq.(9)s. Then, we obtain,

$$\delta\dot{x} = A\lambda e^{\lambda t}, \quad \delta\dot{y} = B\lambda e^{\lambda t} \tag{10.1}$$
$$\delta\ddot{x} = A\lambda^2 e^{\lambda t}, \quad \delta\ddot{y} = A\lambda^2 e^{\lambda t} \tag{10.2}$$

By substituting Eq.(10)s into Eq.(9)s, Eq.(9)s are rewritten as,

$$\begin{bmatrix} \lambda^2 - U_{xx0} & -2n\lambda - U_{xy0} \\ 2n\lambda - U_{xy0} & \lambda^2 - U_{yy0} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = 0 \tag{11}$$

When the solution of Eq.(9)s $(\delta x, \delta y)$ are not zeros, the first matrix on the left-hand side of Eq.(11) is satisfied as,

$$\det \begin{bmatrix} \lambda^2 - U_{xx0} & -2n\lambda - U_{xy0} \\ 2n\lambda - U_{xy0} & \lambda^2 - U_{yy0} \end{bmatrix} = 0 \tag{12}$$

Then, Eq.(12) is rewritten by,

$$\lambda^4 + \left(4n^2 - U_{xx0} - U_{yy0}\right)\lambda^2 + U_{xx0}U_{yy0} - U_{xy0}^2 = 0 \tag{13}$$

Now let us define $4n^2 - U_{xx0} - U_{yy0}$ and $U_{xx0}U_{yy0} - U_{xy0}^2 = 0$ as,

$$a = 4n^2 - U_{xx0} - U_{yy0} \tag{14.1}$$
$$b = U_{xx0}U_{yy0} - U_{xy0}^2 \tag{14.2}$$

Thus, Eq.(13) is rewitten by,

$$\lambda^4 + a\lambda^2 + b = 0 \tag{15}$$

Then, from Eq.(15), λ^2 is,

$$\lambda^2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{16}$$

When the motion of the mass-free body always stays in the vicinity of the libration points, $\lambda^2 < 0$ must be satisfied. By using a and b, this condition can be described as,

$$a > 0 \tag{17.1}$$
$$b > 0 \tag{17.2}$$
$$a^2 - 4b > 0 \tag{17.3}$$

By using the condition of stability, described as Eq.(17)s, the stability of the libration points can be obtained. The results are shown in Table 1.

These solutions are shown in [3], [4].

Table 1: The stability of L_1 through L_5

Equilibrium	L_1, L_2, L_3	L_4, L_5
Stable region	Not exist	$\mu > 24.96$ or $\mu < 0.0401$

4 The Rod-Connected restricted three-body problem

The Rod-Connected Restricted Three-Body Problem (RCRTBP) is described as a model characterized by three bodies; two primary bodies which are connected with each other by a rigid and mass-free stick, and a mass-free bodies. This model is shown in Figure.2.

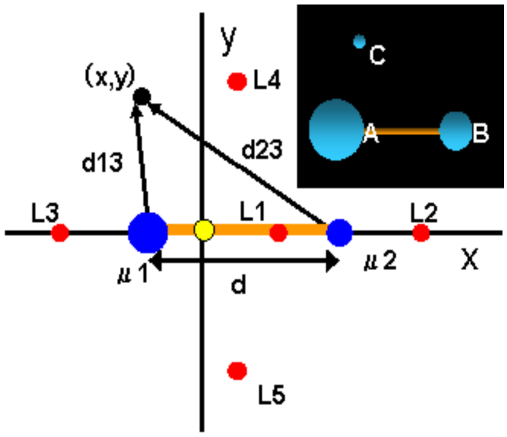

Figure 2: RCRTBP system for the dumbbell shaped body; There are also five equilibriums. Three equilibriums exist along the x axis, and the other two equilibriums are along the perpendicular bisector between the two masses.

4.1 Equation of motion

In this problem, there are two degrees of freedom because of the connection of the primary bodies. In other words, even if only one parameter is determined, the motion of the mass-free bodies can not be described. In this paper, the angular velocity ω, which is different from the mean motion, is newly defined. The way how to set the rotating frame in this system is the same as that in the RTBP system. The equations of motion are described as,

$$\ddot{x} - 2\omega\dot{y} = \tilde{U}_x \tag{18.1}$$
$$\ddot{y} + 2\omega\dot{x} = \tilde{U}_y \tag{18.2}$$

where,

$$\tilde{U} = \frac{\mu_1}{d_{13}} + \frac{\mu_2}{d_{23}} + \frac{1}{2}\omega^2\left(x^2 + y^2\right) \tag{19}$$

These equations of motion are also subject to Eq.(4)s

The difference from the equations of motion in the RTBP system is that the angular velocity is altered from the mean motion n to ω because of increasing the degree of freedom.

4.2 Equilibriums

Like the motion in the RTBP system, at the quilibriums, the condition at the equilibriums is described as,

$$\tilde{U}_x = \tilde{U}_y = 0 \tag{20}$$

From Eq.(20), there are five equilibriums. Three of them lie on the line passing through the primary bodies, and other two points are along the parpendicular bisector between the primary bodies. In this paper, these points are defined as the pseudo libration points and are expressed by from L_1 to L_5. In this paper, from L_1 to L_3 are defined as the pseudo collinear libration points, and L_4 and L_5 are described as the pseudo bisector libration points. These points are shown in Figure.2.

Let us define a new parameter as follows,

$$\eta = \frac{\omega^2}{n^2} \tag{21}$$

By using this parameter, we can obtain the relational expressions of the pseudo collinear libration points as,

$$\frac{\chi^2}{(1-\chi)^2}\frac{\eta(1-\chi)^3 - 1}{\eta\chi^3 - 1} = \mu \tag{22.1}$$
$$\frac{\chi^2}{(1-\chi)^2}\frac{\eta(1+\chi)^3 - 1}{\eta\chi^3 - 1} = -\mu \tag{22.2}$$
$$\frac{\chi^2}{(\chi-1)^2}\frac{\eta(\chi-1)^3 - 1}{\eta\chi^3 - 1} = -\mu \tag{22.3}$$

Eq.(22.1) shows the position of L_1, Eq.(22.2) shows the position of L_2, and Eq.(22.3) shows the position of L_3. Also, the position of the pseudo bisector libration points (x_0, y_0) can obtained as,

$$x_0 = \frac{1-\mu}{2(1+\mu)} \tag{23.1}$$
$$y_0 = \pm\left[\left(\frac{1}{\eta}\right)^{\frac{2}{3}} - \frac{1}{4}\right]^{\frac{1}{2}} \tag{23.2}$$
$$L_4; y_0 > 0, \quad L_5; y_0 < 0$$

From Eq.(22)s and Eq.(23)s, it is found that the position of the pseudo libration points are dependent on two degrees of freedom.

4.3 The stability of the pseudo-libration points

Like the linearized equations of motion in RTBP system, those in the RCRTBP system are described as,

$$\delta\ddot{x} - 2\omega\delta\dot{y} = \tilde{U}_{xx0}\delta x + \tilde{U}_{xy0}\delta y \tag{24.1}$$
$$\delta\ddot{y} + 2\omega\delta\dot{x} = \tilde{U}_{xy0}\delta x + \tilde{U}_{yy0}\delta y \tag{24.2}$$

From Eq(24), the stability conditions in this system are written by,

$$4\omega^2 - \tilde{U}_{xx0} - \tilde{U}_{yy0} > 0 \quad (25.1)$$

$$\tilde{U}_{xx0}\tilde{U}_{yy0} - \tilde{U}_{xy0}^2 > 0 \quad (25.2)$$

$$\left(4\omega^2 - \tilde{U}_{xx0} - \tilde{U}_{yy0}\right)^2 - 4\left(\tilde{U}_{xx0}\tilde{U}_{yy0} - \tilde{U}_{xy0}^2\right) > 0 \quad (25.3)$$

Those stability conditions are obtained by the same processes which are mentioned in the RTBP system. From Eq.(25)s, it is found that while L_2 and L_3 are always unstable in any type of parameters, L_1, L_4, and L_5 have stable regions under the certain condition.

The details about L_1, L_4, and L_5 are reported in the following sections. First, the stability of L_4 and L_5 is discussed. Then, that of L_1 is specified.

4.3.1 The stability at L_4 and L_5

To begin with, let us define θ as the angle between the line joining the two primary bodies and d_{23}. This is shown in Figure.3.

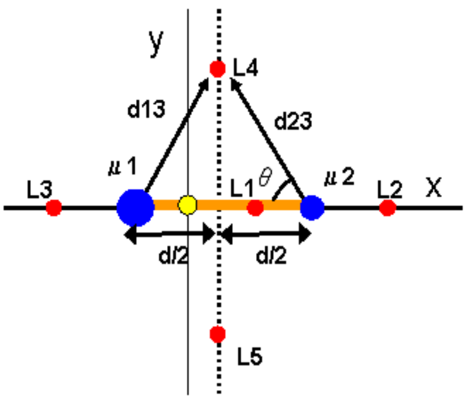

Figure 3: Definition of θ; θ is the angle between the line joining the two primary bodies and d_{23}

Let us consider each equation in Eq.(25)s specifically. First, the left-hand side of Eq.(25.1) is described as,

$$4\eta - (U_{xx0} + U_{yy0}) = \eta > 0 \quad (26)$$

Eq.(26) indicates that Eq.(25.1) is always satisfied at L_4 and L_5.

Second, the left-hand side of Eq.(25.2) is rewritten by,

$$U_{xx0}U_{yy0} - U_{xy0}^2 = \frac{9\mu\eta^{\frac{2}{3}}}{(1+\mu)^2}\left(\left(\frac{1}{\eta}\right)^{\frac{2}{3}} - \frac{1}{4}\right) \quad (27)$$

Then, by substituting the right-hand side of Eq.(27) into Eq.(25.2), the condition of η is described as,

$$\eta < 8 \quad (28)$$

Moreover, using Eq.(23.2), Eq.(28) is rewritten by,

$$\eta = 8\cos^3\theta \quad (29)$$

Eq.(29) also means that

$$\theta \neq 0 \quad (30)$$

At last, by substituting Eq.(29) into Eq.(25.3), we can obtained the third condition defined as,

$$\sin^2\theta\cos^2\theta < \frac{1}{144}\frac{(1+\mu)^2}{\mu} \quad (30)$$

Figure 4 shows the stability at the pseudo bisector libration points. The stable regions are below the boundary curve. In this region, the small fragments can always stay in the vicinity of L_4 and L_5.

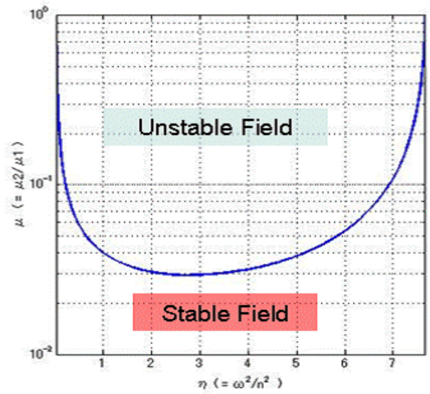

Figure 4: The stability condition at L_4 and L_5; The stable regions are below the boundary curve.

4.3.2 The stability at L_1

The condition of the stability at L_1 is also obtained by the same process mentioned in the above section.

Let us define α as,

$$\alpha = \frac{1}{1+\mu}\left(\frac{\mu}{\chi^3} + \frac{1}{(1-\chi)^3}\right) \quad (32)$$

By using this parameter, the condition at L_1 can be described.

First, from Eq.(22.1), η is described as,

$$\eta = \frac{1}{\chi(\mu+1)-1}\left(\frac{\mu}{\chi^2} - \frac{1}{(1-\chi)^2}\right) \quad (33)$$

By substituting Eq.(32) and Eq.(33) into Eq.(25)s, Eq.(25)s are rewritten by,

$$2\eta - \alpha > 0 \quad (34\text{-}1)$$

$$\eta - \alpha > 0 \quad (34\text{-}2)$$

$$\eta < 9/8\alpha \quad (34\text{-}3)$$

Eq.(34-1) is obtained from Eq.(25.1), Eq.(34-2) is procured by using Eq.(25.2), and Eq.(34-3) is the equivalent condtion of Eq.(25.3).

Therefore, from Eq.(34)s, the stability condition is described as,

$$\alpha < \eta < \frac{9}{8}\alpha \tag{24}$$

Figure 5 shows the stability at L_1. The stable regions are between the two boundary curves. Also, we indicated in this figure the stability condition in RTBP system. The line of $\eta = 1$ indicates it. According to this figure, L_1 in RTBP system is always unstable.

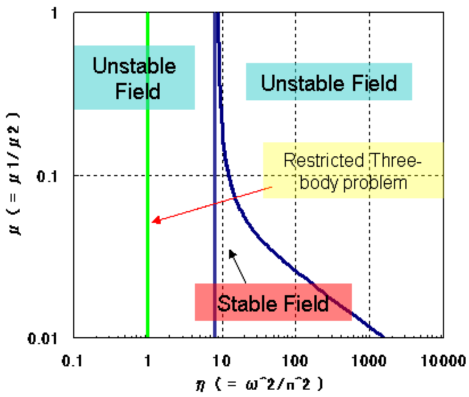

Figure 5: The stability condition at L_1; The stable regions are between the two boundary curve

5 Conclusion

This paper focused on the stability in the Rod-Connected Restricted Three Body Problem for the dumbbell shaped body. The stability of this model was specified by applying the Restricted Three Body Problem. By using the fact that there are no forces which affect the motion of the mass-free body at the equilibriums, five pseudo libration points were obtained. Three of them which were defined as from L_1 to L_3 lie on the x axis, and other two equilibriums which were expressed by L_4 and L_5 are along the perpendicular bisector between the primary bodies. Then, by linearizing the equation of motion, the linear stability of the motion of the mass-free body in the vicinity of the pseudo libration points was investigated. According to this investigation, L_2 and L_3 are always unstable, L_1, L_4, and L_5 have stable regions. In this problem, because there are two degrees of freedom in the Rod-Connected Restricted Three Body Problem, the arbitrary property of the position of those pseudo libration points occurs. For example, L_4 and L_5 are not necessary to be at the apex of an equilateral triangle, mentioned in the Restricted Three Body Problem. Thus, in this paper, the position of those pseudo libration points and the stability at them were indicated by three parameters: η, χ, and μ.

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