Transfer between Periodic Orbits along a Circular Orbit

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Abstract: Initial conditions of the nonlinear relative dynamics along a circular orbit, which yield periodic solutions, are characterized, and a formation problem based on them is formulated. Using the property of null controllability with vanishing energy of the Hill-Clohessy-Wiltshire equations, $L_1$ suboptimal feedback controllers are designed. By simulation results, the effectiveness of the designed controllers are illustrated.

1 Introduction

In this paper, periodic orbits of the HCW system are replaced by those of the original nonlinear relative dynamics, and the corresponding formation problem is considered. As in [1], $L_1$ suboptimal controllers are sought. As preliminaries, initial conditions of the nonlinear dynamics which yield periodic solutions are characterized in terms of eccentric orbits with semimajor axis equal to the radius of the circular orbit. In this paper, coplanar orbits are discussed. Using these initial conditions, the formation problem is formulated, and suboptimal controllers are designed through the algebraic Riccati equation of the LQR theory. Using an extension of [2] to discrete time systems [3, 4], the NCVE property is also proved in [1] for the Tschauner-Hempel equations, which are the linearized equations of the relative dynamics along an eccentric orbit. It is used in [5] for the relative orbit transfer problem by impulsive control.

To adjust given initial conditions of the nonlinear relative dynamics, by a single impulse, to those of a periodic solution is called the initialization problem, and is studied in [6] in the case of an eccentric orbit using a different characterization of the initial conditions of periodic solutions.

The paper is organized as follows. Section 1 is Introduction. Section 2 introduces the equations of the relative motion, and their state space form. Section 3 gives the characterization of initial conditions of the nonlinear relative dynamics corresponding to periodic solutions, and Section 4 formulates the formation problem. Section 5 discusses the design of feedback controllers, and finally Section 6 collects simulation results on three examples.

2 Equations of Relative Motion

Consider two satellites subject to the central gravity field of the Earth, one of which is flying on a given circular orbit of radius $R_0$ and is referred to as a leader, and the other flying nearby is referred to as a follower. The relative motion of the latter with respect to the former is given by Newton’s equations of motion as follows

$$\dot{x} = 2n\dot{y} + n^2(R_0 + x) - \frac{\mu}{R_0^3}(R_0 + x) + u_x,$$
$$\dot{y} = -2nx + n^2y - \frac{\mu}{R_0^3}y + u_y,$$
$$\dot{z} = -\frac{\mu}{R_0^3}z + u_z,$$

where the coordinate system $(x, y, z)$ is fixed at the center of mass of the leader, $x$, $y$, and $z$-axes are along the radial direction, the flight direction of the leader, and the normal direction respectively, $\mu$ is the gravitational parameter of the Earth, $n = (\mu/R_0^3)^{1/2}$ the orbit rate of the leader, $\dot{R} = [(R_0 + x)^2 + y^2 + z^2]^{1/2}$, and $u_x$, $u_y$, and $u_z$ are the control accelerations. The linearized equations around the origin are given by

$$\ddot{x} = 2n\dot{y} + 3n^2x + u_x,$$
$$\ddot{y} = -2nx + u_y,$$
$$\ddot{z} = -n^2z + u_z,$$

which are known as Hill-Clohessy-Wiltshire (HCW) equations. The in-plane motion (2), (3) is independent of the out-of-plane motion (4). In this paper the in-plane motion is considered. The solution of HCW equations is periodic if and only if the CW condition $g_0 = -2nx_0$ holds. In this case the in-plane motion is given by

$$x(t) = a\cos(nt + \alpha),$$
$$y(t) = d - 2a\sin(nt + \alpha),$$

where $a$ is the initial relative radius, $d$ the distance of the follower from the origin, and $\alpha$ the initial relative phase.

The normal direction $\hat{z}$ of the central gravity field is perpendicular to both the in-plane orbit of the follower, and the orbit of the leader, and is aligned with the relative motion of the follower.

The out-of-plane motion (4) is given by

$$\dot{z} = -\frac{\mu}{R_0^3}z + u_z,$$
where 

\[ a = \left(3x_0 + 2\dot{y}_0/n\right)^2 + (\dot{x}_0/n)^2 \right)^{1/2}, \quad d = y_0 - 2\dot{x}_0/n, \]
\[ \cos \alpha = -(1/a) \left(3x_0 + 2\dot{y}_0/n\right), \quad \sin \alpha = -\dot{x}_0/(na). \] 

It forms an ellipse 

\[ \frac{x^2}{a^2} + \frac{(y - d)^2}{(2a)^2} = 1 \] 

in \((x, y)\) plane, and is useful for formation as well as proximity operations such as inspection and repair.

Defining 

\[ x = [x \ y \ \dot{x} \ \dot{y}]^T, \quad u = [u_x \ u_y]^T, \]

the HCW equations in state space form become

\[ \dot{x} = Ax + Bu, \] 

where 

\[ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 2n \\ 0 & -2n & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \]

and (1) is rewritten in a semilinear form

\[ \dot{x} = Ax + Bu + g(x), \] 

where

\[ g(x) = \begin{bmatrix} 0 \\ 0 \\ -3n^2x - (R_0 + x)(\mu/R^3 - n^2) \\ -y(\mu/R^3 - n^2) \end{bmatrix}. \]

3 Periodic Solutions of Relative Dynamics and Their Initial Conditions

Although periodic solutions (5) of the HCW equations are useful for formation flight and various other operations, they are not free motions of the nonlinear equations (1), and hence control efforts are required to maintain them. In this section initial conditions which constitute a periodic free motion of (1) will be found. Assume that the follower is in an eccentric orbit \( \Gamma = (A_0, e) \) in the inertial frame, where \( A_0 \) is the semimajor axis and \( e \) is the eccentricity vector. Let \((X, Y, Z)\) be its perifocal reference frame whose origin is the center of the Earth and the \(X\) axis is in the direction of the eccentricity vector \( e \). Assume first that the orbit is coplanar with the circular orbit of the leader shown in Fig. 4, where \( R, R_0 \) are the position vectors of the follower and the leader respectively, and \( r \) is the relative position vector. If the \(Z\) coordinate of the perigee is \( A_0 - a \), then the eccentricity \( e = |\mathbf{e}| = a/A_0 \). Let \( t_0 \) be the time at which the leader is in the direction of \( e \) (on the \(X\) axis). Let \( \theta(t) \) be the true anomaly of the follower, and set \( \theta_0 = \theta(t_0) \). Below the initial conditions of the relative motion (1) at \( t_0 \) will be calculated. Recall that the relative motion of the follower is periodic if and only if \( A_0 = R_0 \), because both periods of the circular and eccentric orbits are the same.

Note that \( r(t) \) in the \((X, Y, Z)\) coordinate system is given by

\[ r(t) = R(t) - R_0(t) \]

\[ = \begin{bmatrix} R(t) \cos \theta(t) - R_0 \cos n(t - t_0) \\ R(t) \sin \theta(t) - R_0 \sin n(t - t_0) \end{bmatrix}, \]

where \( R = |R| \) is given by

\[ R(t) = p/(1 + e \cos \theta(t)), \quad p = A_0(1 - e^2). \]

To find the relative velocity of the follower, the lemma below is useful.

Lemma 3.1.

\[ p \dot{\theta}(t) = \left(\mu/p^{1/2}(1 + e \cos \theta(t))\right)^2, \]
\[ p \frac{d}{dt} \cos(\theta(t) + \beta) \]
\[ = -\left(\mu/p^{1/2}(\sin(\theta(t) + \beta) + e \sin \beta), \right. \]
\[ p \frac{d}{dt} \sin(\theta(t) + \beta) \]
\[ = -\left(\mu/p^{1/2}(\cos(\theta(t) + \beta) + e \cos \beta). \right. \]
The first equation follows from the Kepler’s second law concerning the area velocity, while other two equalities are obtained by direct differentiation.

Note that the angular velocity of the rotating frame \((x, y, z)\) is \(\Omega = [0 \ 0 \ n]^T\) and that the derivatives of \(r(t)\) in the \((X, Y, Z)\) and \((x, y, z)\) frames denoted respectively by \(\dot{r}\) \((dr/dt)\) and \(\delta r/\delta t\) are related by the following equation

\[
\dot{r} = \frac{dr}{dt} = \frac{\delta r}{\delta t} + \omega \times r.
\]

Hence the relative velocity \(\delta r/\delta t\) expressed in the \((X, Y, Z)\) coordinate system is given by

\[
\frac{\delta r}{\delta t}(t) = \dot{r} - \omega \times r(t) = \dot{R} - \omega \times R(t)
\]

\[
= \left[ \begin{array}{c}
-\frac{n_A}{\sqrt{1-e^2}} \sin \theta(t) + nR(t) \sin \theta(t) \\
\frac{n_A}{\sqrt{1-e^2}} \cos \theta(t) + e - nR(t) \cos \theta(t)
\end{array} \right],
\]

(12)

where \(\delta R_0(t)/\delta t = 0\), Lemma 3.1 with \(\beta = 0\), and \((\mu/p)^{1/2} = n_0A_0/(1-e^2)^{1/2}\) with \(n_0 = (\mu/A_0)^{1/2}\) are used. Thus the relative velocity at \(t_0\) is

\[
\frac{\delta r}{\delta t}(t_0) = \left[ \begin{array}{c}
-\frac{n_A}{\sqrt{1-e^2}} \sin \theta_0 + \frac{n_A}{\sqrt{1-e^2}} \sin \theta_0 \\
\frac{n_A}{\sqrt{1-e^2}} \cos \theta_0 + e - \frac{n_A}{\sqrt{1-e^2}} \cos \theta_0
\end{array} \right].
\]

(13)

Because \(\delta r/\delta t\) in two coordinate systems \((X, Y, Z)\) and \((x, y, z)\) coincide at \(t_0\), (13) is also the relative velocity in the \((x, y, z)\) coordinate system. Hence in view of (11) and (13), the initial conditions at \(t_0\) of (1) are given by

\[
\begin{bmatrix}
x(t_0) \\
y(t_0)
\end{bmatrix} = \begin{bmatrix}
\frac{A_0(1-e^2)}{1+e \cos \theta_0} \cos \theta_0 - R_0 \\
\frac{A_0(1-e^2)}{1+e \cos \theta_0} \sin \theta_0
\end{bmatrix},
\]

(14)

\[
\begin{bmatrix}
\dot{x}(t_0) \\
\dot{y}(t_0)
\end{bmatrix} = \begin{bmatrix}
-\frac{n_A}{\sqrt{1-e^2}} \sin \theta_0 + \frac{n_A}{\sqrt{1-e^2}} \sin \theta_0 \\
\frac{n_A}{\sqrt{1-e^2}} \cos \theta_0 + e - \frac{n_A}{\sqrt{1-e^2}} \cos \theta_0
\end{bmatrix}.
\]

(15)

The initial conditions can be directly derived from \(r(t)\) in the system \((x, y, z)\) given by

\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix}
R(t) \cos[\theta(t) - n(t-t_0)] - R_0 \\
R(t) \sin[\theta(t) - n(t-t_0)]
\end{bmatrix}.
\]

In fact \(t = t_0\) yields the initial position (14), and the derivative at \(t_0\) gives the initial velocity.

The initial conditions (14) and (15) with \(A_0 = R_0\) and \(n_0 = n\) yield a periodic solution of (1), and the following theorem holds.

\section{Formation Problem}

In this section, a formation problem based on periodic relative orbits is formulated. First relative orbits are characterized by some parameters. Let \((X_0, Y_0, Z_0)\) be an inertial reference frame such that the \((X_0, Y_0)\) plane coincides with the orbit plane of the leader. As seen in the previous section, a coplanar orbit of the follower is denoted by \(\Gamma = (A_0, e)\), and the relative motion is determined by \(t_0\) and \(\theta_0\). Thus it is denoted by \(\gamma = (A_0, e, t_0, \theta_0)\). The formation problem in this paper is described as follows. For a given initial relative orbit \(\gamma = (A_0, e, t_0, \theta_0)\) and a final orbit \(\gamma_f = (R_f, e_f, t_{f0}, \theta_{f0})\), find a feedback control strategy such that the solution of (9) starting form the initial orbit tracks \(\gamma_f\) asymptotically. The performance index of the controller is the \(L_1\) norm, which represents the total velocity change, and hence fuel consumption. If, in particular, the initial orbit is periodic and \(\gamma = (R_0, e, t_0, \theta_0)\), it is a re-formation problem. Note that the relative dynamics of the follower is given by (9) so that

\[
\dot{x} = Ax + Bu + g(x), \quad x(t_0) = x_0.
\]

In order to track \(\gamma_f\), a virtual satellite is introduced, which follows the free motion of (9) i.e.,

\[
\dot{x}_f = Ax_f + g(x_f), \quad x_f(t_{f0}) = x_{f0},
\]

where \(x_{f0}\) is determined by \(\gamma_f\). Note that two initial conditions are given at different times \(t_0\) and \(t_{f0}\).

Without loss of generality, the inequality \(t_0 > t_{f0}\) is assumed below. For control purpose, it is useful to find the state \(x_f(t_0)\) of the virtual satellite at \(t_0\). The relative position and velocity of the virtual satellite
at $t$ in the perifocal reference frame of the final orbit $\Gamma = (R_0, e_f)$ are given by

$$r_f(t) = R_f(t) - R_0(t)$$

$$= \begin{bmatrix} R_f(t) \cos(\theta_f(t) - n(t_0 - t_{f0})) \\ R_f(t) \sin(\theta_f(t) - n(t_0 - t_{f0})) \\ -R_0 \cos n(t - t_0) \\ -R_0 \sin n(t - t_0) \end{bmatrix},$$

where

$$R_f(t) = R_0(1 - e_f^2)/(1 + e_f \cos \theta_f(t)).$$

Hence as in the previous section,

$$r_f(t_0) = \begin{bmatrix} R_f(t_0) \cos(\theta_f(t_0) - n(t_0 - t_{f0})) - R_0 \\ R_f(t_0) \sin(\theta_f(t_0) - n(t_0 - t_{f0})) \\ -R_0 \cos n(t_0 - t_{f0}) \\ -R_0 \sin n(t_0 - t_{f0}) \end{bmatrix}.$$ (19)

and

$$\frac{\delta r_f(t_0)}{\delta t} = \dot{r}_f(t_0) - \omega \times r_f(t_0)$$

$$= \begin{bmatrix} -\frac{nR_0}{\sqrt{1 - \epsilon_f^2}} [\sin(\theta_f(t_0) - n(t_0 - t_{f0}))] \\ \frac{nR_0}{\sqrt{1 - \epsilon_f^2}} [\cos(\theta_f(t_0) - n(t_0 - t_{f0}))] \\ -e_f \sin n(t_0 - t_{f0}) + nR_f(t_0) \sin(\theta_f(t_0)) \\ +e_f \cos n(t_0 - t_{f0}) + nR_f(t_0) \cos(\theta_f(t_0)) \end{bmatrix}.$$ (20)

where Lemma 3.1 with $\beta = -n(t_0 - t_{f0})$ is used. $\theta_f(t_0)$ is determined by the Kepler’s time equation

$$n(t_0 - t_p) = E_f(t_0) - e_f \sin E_f(t_0)$$

and the relation

$$\cos \theta_f(t_0) = \frac{\cos E_f(t_0) - e_f}{1 - e_f \cos E_f(t_0)}.$$

where $E_f$ is the eccentric anomaly, and the perigee passage time $t_p$ is given by

$$n(t_{f0} - t_p) = E_f(t_{f0}) - e_f \sin E_f(t_{f0}),$$

$$\cos \theta_f(t_{f0}) = \frac{\cos E_f(t_{f0}) - e_f}{1 - e_f \cos E_f(t_{f0})}.$$ 

The initial condition $x_f(t_0)$ is then obtained by rearranging (19) and (20).

5 $L_1$ Suboptimal Feedback Controllers

To design feedback controllers which steer the follower asymptotically to the final orbit, the error dynamics

$$\dot{e} = Ae + Bu + g(x) - g(x_f)$$ (21)

is introduced, where $e = x - x_f$. As a performance index, the $L_1$ norm of a controller, which represents the fuel consumption, is employed. First recall that the linear part of (21) is the suboptimal feedback controller.

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are given respectively by
\[ h, R, \mu \] from the same positions and produce periodic solutions of the nonlinear system (21). Designed. Moreover, formation problems using periodic orbits of the HCW system and the nonlinear system (21) are compared, and similar performance indices are obtained. Hence no extra efforts are needed to replace periodic orbits of the HCW by those of the nonlinear system (1).

### 6 Simulation Results

For numerical simulations, a circular orbit of the leader of height \( h_c = 500 \) km is considered. Then the period of this orbit is \( T = 5677 \) s, and the orbit rate \( n = 1.1068 \times 10^{-3} \) rad/s see Table 1, where the radius of the Earth \( R_e \) and the gravitational constant of the Earth \( \mu \) are also given. Let \((i, j, k)\) be the basis vectors of the inertial frame \((X_0, Y_0, Z_0)\).


Consider two periodic relative orbits denoted by \( \gamma = ((50/R_0)i, 0, 0) \) and \( \gamma_f = ((5/R_0)i, 0, 0) \), where \( R_0 = R_e + 500 \), and consider the re-formation from \( \gamma \) to \( \gamma_f \). In this case, \((X_0, Y_0, Z_0)\) is also the perifocal reference system of the two orbits. The \( X_0 \) coordinates of the perigee are \( R_0 - a \) with \( a = 50 \), and 5 respectively. Moreover, the leader, follower and virtual satellite are initially on the \( X_0 \) axis so that \( t_0 = t_{f0} = 0 \).

Using (16) and (17), the initial conditions of the nonlinear system (9) and (18) are given respectively by

\[
\begin{align*}
x_0 &= \begin{bmatrix} -50.0000 & 0.00000 & 0.11064 & 0.00000 \end{bmatrix}, \\
x_{f0} &= \begin{bmatrix} -5.00000 & 0.00000 & 0.01104 & 0.00000 \end{bmatrix}.
\end{align*}
\]

Initial conditions of the HCW system (8) which start from the same positions and produce periodic solutions are given respectively by

\[
\begin{align*}
x^H_0 &= \begin{bmatrix} -50.0000 & 0.00000 & 0.11044 & 0.00000 \end{bmatrix}, \\
x^H_{f0} &= \begin{bmatrix} -5.00000 & 0.00000 & 0.01104 & 0.00000 \end{bmatrix}.
\end{align*}
\]

### Table 1: Constants and common parameters

<table>
<thead>
<tr>
<th>Constants</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_e )</td>
<td>6378.136 km</td>
</tr>
<tr>
<td>( \mu )</td>
<td>398601 km³/s²</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Common parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_c )</td>
<td>500 km</td>
</tr>
<tr>
<td>( n )</td>
<td>1.1068×10⁻³ rad/s</td>
</tr>
<tr>
<td>( T )</td>
<td>5676 s</td>
</tr>
</tbody>
</table>

They correspond to the two concentric relative orbits (7) with \( d = 0 \) and \( a = 50 \) km, 5 km, respectively. The initial phases of the follower and the virtual satellite are \( \alpha = \pi \). To design a feedback controller, \( Q \) and \( R \) in the ARE (23) are assumed to be diagonal, i.e., \( Q = \text{diag}(q_i) \) and \( R = 10^r I \) with \( q_i = 1.0 \times 10^{-9}, i = 1, 2, \) and \( q_i = 0, i = 3, 4 \). The controller (24) is applied to the nonlinear system (9). The settling time \( T_s \) and \( L_1 \) and \( L_2 \) norms of the feedback controller against parameter \( r \) are plotted in Figs. 3 – 5. The settling time is defined as the first time after which inequalities \(|u_x|, |u_y| < 1.0 \times 10^{-6} \text{m/s}^2\) are satisfied. The \( L_2 \) norm approaches zero as \( r \) tends to infinity, confirming the NCVE property, whereas the \( L_1 \) norm approaches a positive constant. For comparison, the free motion of the virtual satellite (18) is replaced by the linear equation

\[
\dot{x}^H_f = A x^H_f, \quad x^H_f(t_{f0}) = x^H_{f0}.
\]

In this case the controller asymptotically brings the follower to the periodic orbit of the HCW system given above. All plots in two cases are almost identical.

To assure a reasonable settling time, \( r = 4 \) giving \( T_s = 17527 \) s is chosen. In this case the \( L_1 \) norm of the controller is 43.917 Nms and the controlled trajectory of the follower is given in Fig. 6. Infima of the \( L_1 \) norms are given in Table 2.

In the relative orbit transfer problem for the HCW system [1], the nonlinear system (21) is replaced by the HCW system

\[
\dot{e} = Ae + Bu.
\]

The minimum cost of the LQR problem associated with this is given by \( e_0^T X e_0 \), and further minimization with respect to the initial condition \( e_0 \) was performed. When the follower is allowed to stay in the initial orbit and to choose the initial time freely, this determines the best initial position to start control, which minimizes the \( L_2 \) norm of the controller. Then the \( L_1 \) norm is also minimized. In Fig. 7, \( L_1 \) norms are plotted against the initial phase \( 0 \leq \alpha \leq 2\pi \). The additional plot denoted by “HCW” gives the \( L_1 \) norm of the feedback for the linear system (25). The best initial positions in the nonlinear- and linear cases are respectively \( \alpha = 3.92 \) and \( \alpha = 3.97 \), which are almost identical and thus the linear feedback controller is effective for the nonlinear system (9).

### Table 2: Fuel consumption

<table>
<thead>
<tr>
<th>Reference Periodic Orbit</th>
<th>plant ( L_1 ) norm [Nms]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>Eq. (9)</td>
</tr>
<tr>
<td>Nonlinear</td>
<td>Eq. (9)</td>
</tr>
</tbody>
</table>

Reference Periodic Orbit plant

43
2
917
10
- and

9
2
- and

3
9
3
776

Norms are plotted against

\(|u_x|, |u_y| < 1.0 \times 10^{-6} \text{m/s}^2\) are satisfied. The

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Using (19) and (20),

\[ x \text{ at } \text{on the negative part of the } \]

The final orbit is \( \gamma (t_f) = \left( (50/R_0) t_f, 0, 0, 2 \right) \).

The parameters are set \( q_i = 1.0 \times 10^{-9} \) for \( i = 1, 2, q_i = 0 \) for \( i = 3, 4, \) and \( r = 4. \)

In Fig. 8, two orbits \( \gamma, \gamma_f \) and the controlled trajectory are given. The \( L_1 \) norm of the controller is 115.12 Nms, and the settling time \( T_s = 15008 \) s.

**Example 6.2. Coplanar formation: general case.**

Consider the initial and final two orbits given by \( \gamma = ((50/R_0) t, 0, 0.02) \) and \( \gamma_f = ((50/R_0) t_f, -T/4, 0) \). The angle between \( e_0 \) and \( e_f \) is \( \pi/2 \). The perifocal reference system of the initial orbit is \( (X_0, Y_0, Z_0) \). At time \( t_0 = 0 \), the leader is on the \( X_0 \) axis, i.e., in the direction of \( e_0 \). At time \( t_f = -T/4 \), the leader was on the negative part of the \( Y_0 \) axis, which is the direction of \( e_f \). Hence the perifocal reference system of the final orbit is \( (-Y_0, X_0, Z_0) \). Then by (16) and (17), initial conditions of the follower and the virtual satellite at \( t_0 = 0 \) and \( t_f = -T/4 \) are given respectively by

\[
x_0(t_0 = 0) = 
\begin{bmatrix}
-51.35766 \\
1.36747 \\
0.00110 \\
0.00000
\end{bmatrix},
\]

\[
x_f(t_f = -T/4) = 
\begin{bmatrix}
-0.50000 \\
0.00000 \\
0.00110 \\
0.00000
\end{bmatrix}.
\]

Using (19) and (20), \( x_f(t_0 = 0) \) is given by

\[
x_f(t_0 = 0) = 
\begin{bmatrix}
-0.37611 \\
99.99828 \\
0.05522 \\
-0.00017
\end{bmatrix}.
\]

The parameters are set \( q_i = 1.0 \times 10^{-9} \) for \( i = 1, 2, q_i = 0 \) for \( i = 3, 4, \) and \( r = 4. \). In Fig. 8, two orbits \( \gamma, \gamma_f \) and the controlled trajectory are given. The \( L_1 \) norm of the controller is 115.12 Nms, and the settling time \( T_s = 15008 \) s.