Attitude Control of a Satellite Based on Euler Angle Representation

オイラー角を用いた宇宙機の姿勢追従制御

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Abstract

In future missions such as inspection, repair or removal of disabled spacecrafts, control of spacecraft attitude becomes an important issue. In particular, it is necessary to construct a new control strategy whereby we use less amount of energy to control spacecrafts. First, we derive and linearize equations of relative motion between the desired attitude motion and the present attitude motion, which are based on the Euler angle representation. We then design linear feedback controllers based on the linearized equations and nonlinear feedback controllers based on the feedback linearization. We consider the $L_1$ norm of the feedback controllers as the main performance index, because it is proportional to fuel consumption.

1 Introduction

In future missions such as inspection, repair or removal of disabled spacecrafts, control of spacecraft attitude becomes an important issue. In particular, it is necessary to construct a new control strategy whereby we use less amount of energy to control spacecrafts. Our goal is to reduce the $L_1$ norm of the input torque, which is proportional to the amount of energy to control rotational motion of a spacecraft. However, currently, we do not have an efficient method to do so. To design feedback controllers, we employ the linear quadratic regulator (LQR) theory together with the property of null controllability with vanishing energy (NCVE) [1, 2, 3].

NCVE is a property that any initial state can be steered to the origin with arbitrarily small amount of control energy in the sense of $L_2$. Intuitively one expects that the value of $L_1$ norm gets smaller as the value of $L_2$ is reduced. We shall examine by simulations the behavior of $L_1$ norm.

In this paper, we shall use Euler angle representation to describe the attitude of spacecrafts. We consider the relative motion between a chaser satellite and a target satellite. We will consider two types of motions of the target spacecraft: pure spin and flat spin.

2 Relative Rotational Equations of Motion

Let $R_C$ be a body-fixed reference frame of the chaser satellite with basis vectors $(c_1, c_2, c_3)$ and $R_T$ the reference frame of the target with basis vectors $(t_1, t_2, t_3)$, respectively. A general rotational motion is given by the kinematics and dynamics differential equations [4]

$$
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix} = \frac{1}{\cos \theta_3}
\begin{bmatrix}
\cos \theta_3 & -\cos \theta_1 \sin \theta_3 & \sin \theta_1 \sin \theta_3 \\
0 & \cos \theta_1 & -\sin \theta_1 \\
0 & \sin \theta_1 \cos \theta_3 & \cos \theta_1 \cos \theta_3
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix},
$$

$$J\ddot{\omega} + \Omega J\omega = M,$$

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where $\Omega$ is a skew-symmetric matrix given by

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$  

(3)

To derive the kinematics equation (1), we take three successive body-axis rotations, $C_1(\theta_1) \leftarrow C_3(\theta_3) \leftarrow C_2(\theta_2)$, where $C_i(\theta_i)$ denotes the direction cosine matrix of an elementary rotation about the $i$th axis of the target by an angle $\theta_i$.

Next, we introduce rotational motion of the chaser relative to that of the target. The relative angular velocity vector $\omega^{C/T}$ is given as $\omega^{C/T} = \omega^C - C^{C/T} \omega^T$. Substituting the relative Euler angle and the relative angular velocity into equations (1) and (2), respectively, we obtain the equations of relative motion

$$\begin{bmatrix} \dot{\theta}_{1}^{C/T} \\ \dot{\theta}_{2}^{C/T} \\ \dot{\theta}_{3}^{C/T} \end{bmatrix} = \frac{1}{\cos \theta_3^{C/T}} \begin{bmatrix} \cos \theta_3^{C/T} \sin \theta_1^{C/T} & -\sin \theta_3^{C/T} \sin \theta_1^{C/T} & \sin \theta_3^{C/T} \\ \cos \theta_3^{C/T} \sin \theta_2^{C/T} & -\sin \theta_3^{C/T} \sin \theta_2^{C/T} & \sin \theta_3^{C/T} \\ \sin \theta_3^{C/T} & -\cos \theta_3^{C/T} & 0 \end{bmatrix} \begin{bmatrix} \omega_1^{C/T} \\ \omega_2^{C/T} \\ \omega_3^{C/T} \end{bmatrix}$$

(4)

$$J(\dot{\omega}^{C/T} - \Omega^{C/T} C^{C/T} \omega^{T} + C^{C/T} \dot{\omega}^{T}) + (\omega^{C/T} + C^{C/T} \omega^{T}) \times J(\omega^{C/T} + C^{C/T} \dot{\omega}^{T}) = M$$

(5)

where $\theta^{C/T}$ is the Euler angle of the reference frame of $RC$ relative to the reference frame of $RT$, $\omega^{C/T}$ is the relative angular velocity, and $C^{C/T}$ is the direction cosine matrix of the reference frame of $RC$ relative to $RT$.

Linearizing equations (4) and (5) at the origin, $\theta^{C/T} = [0, 0, 0]$ and $\omega^{C/T} = [0, 0, 0]$, which represents the identical state of the relative motion, we obtain the linearized system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

(6)

where

$$x = \begin{bmatrix} \theta_{1}^{C/T} & \theta_{2}^{C/T} & \theta_{3}^{C/T} & \omega_{1}^{C/T} & \omega_{2}^{C/T} & \omega_{3}^{C/T} \end{bmatrix}^t,$$

(7)

and

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ k_1 (\omega_1^T - \omega_2^T) & \omega_1^T \omega_2^T (k_1 + k_3) & -\omega_1^T \omega_2^T (k_1 + k_2) & 0 & (k_1 + 1) \omega_3^T & (k_1 - 1) \omega_3^T \\ -\omega_3^T \omega_2^T (k_2 + k_3) & k_2 (\omega_1^T - \omega_2^T) & \omega_2^T \omega_3^T (k_1 + k_2) & 0 & (k_2 - 1) \omega_3^T & (k_2 + 1) \omega_3^T \\ \omega_2^T \omega_3^T (k_2 + k_3) & -\omega_2^T \omega_3^T (k_1 + k_3) & k_3 (\omega_1^T - \omega_2^T) & 0 & (k_3 + 1) \omega_2^T & (k_3 - 1) \omega_2^T \end{bmatrix},$$

(8)

$$B = \begin{bmatrix} O_{6\times 1} \\ \text{J}^{-1} \end{bmatrix}.$$  

(9)

### 3 NCVE for Axisymmetric Spacecraft

In this paper, we assume that the spacecrafts are axisymmetric. The axis of symmetry is the third axis, which means the principal moments of inertia around the first and second axis are equal: $J_1 = J_2 = J$. Then $k_1 = -k_2 = k$ and $k_3 = 0$. The motion of the target satellite is described simply by

$$\dot{\omega}_1^T = \mu \omega_2^T, \quad \dot{\omega}_2^T = \mu \omega_1^T, \quad \dot{\omega}_3^T = 0$$

(11)

where $\mu = k \omega_3^T$, and $\omega_3^T$ (constant value) is a spin rate around the third axis.

Substituting $k_1 = -k_2 = k$ and $k_3 = 0$ into equation (6), the system matrix becomes

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ k_1 (\omega_1^T - \omega_2^T) & k_2 (\omega_1^T - \omega_2^T) & 0 & 0 & (k_1 + 1) \omega_3^T & (k_1 - 1) \omega_3^T \\ k_2 (\omega_1^T - \omega_2^T) & -k_2 (\omega_1^T - \omega_2^T) & 0 & 0 & (k_2 - 1) \omega_3^T & (k_2 + 1) \omega_3^T \\ -k_3 (\omega_1^T - \omega_2^T) & k_3 (\omega_1^T - \omega_2^T) & 0 & 0 & (k_3 + 1) \omega_2^T & (k_3 - 1) \omega_2^T \end{bmatrix}.$$  

(12)
Recall that NCVE is a property for a linear system that any state of the system can be steered to the origin with an arbitrarily small amount of energy in the $L_2$ sense, as the time duration becomes large. A system $(A, B)$ is NCVE(CVE) if and only if $(A, B)$ is controllable and all eigenvalues of $A$ have nonpositive real parts [3]. The second condition can be replaced by that for which the maximal solution of the singular algebraic Riccati equation is zero.

**Theorem 1.** The system is NCVE when the target is rotating around the third axis.

**Proof.** Let us assume $\omega = [0, 0, \omega_3 (const.)]$, which indicates that the motion of the target is in pure spin. Then the system matrix $A$ is rewritten as

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
k(\omega_3^2) & 0 & 0 & 0 & (k + 1)\omega_3^2 & 0 \\
0 & k(\omega_3^2) & 0 & -(k + 1)\omega_3^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{13}$$

The set of eigenvalues of the system matrix (13) is

$$\sigma(A_{pur}) = \{0\text{(double)}, \pm i\omega_3, \pm ik\omega_3\}. \tag{14}$$

**Theorem 2.** The system is NCVE when the target is rotating around the first or second axis.

**Proof.** Let us consider $\omega = [0, \omega_2 (const.), 0]$, which indicates that the motion of the target is in flat (or transverse) spin. Then the system matrix $A$ becomes

$$A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-k(\omega_2^2) & 0 & 0 & 0 & (k - 1)\omega_2^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k\omega_2^2 & 0 & 0 & 0
\end{bmatrix}. \tag{15}$$

The set of eigenvalues of the system matrix (15) is

$$\sigma(A_{tra}) = \{0\text{(quadruple)}, \pm i\omega_1\}. \tag{16}$$

### 4 Simulation Results

In this paper, we design feedback controllers by using the linear-quadratic regulator (LQR) theory, which provides the optimal input $u$ to minimize the quadratic cost function

$$J = \int_0^\infty (x^T Q x + R u^T u) dt \tag{17}$$

where

$$Q = Q \text{diag}\{200, 200, 200, 1, 1, 1\} \tag{18}$$

We set the inertia matrix as

$$J = \text{diag}\{100, 100, 50\}[kgm^2] \tag{19}$$

Thus, $k_1 = -k_2 = k = 0.25$ and $k_3 = 0$.

In the first case, we assume that the target rotates around the (second) axis i.e. flat spin. The initial condition of the target is $\theta^T = [\pi/20, 0, 0]$, and $\omega^T = [0, \pi/9, 0]$, and the initial condition of the chaser is $\theta^C = [0, 0, \pi/12]$, and $\omega^C = [0, \pi/10, 0]$.

The chaser is also initially spinning around the second axis.

Simulation results with $Q = 1$ and $R = 100$ are shown in Fig. 1. The Euler angles are given on the left side, and the angular velocities on the right side, respectively. The first row is the motion of the target, the second row is that of the chaser, the third row is the relative motion which is given by the nonlinear equation, and the fourth row is
the relative motion given by the linearized equation. In Fig. 2, we plotted the settling time, the $L_2$ and $L_1$ norms of
the input torque as function of $R$. The results are shown in Fig. 2.

We find that the settling time decreases as $R$ grows, since the second term of the cost function (17) gets dominant,
and that the $L_2$ norm tends to zero. The latter observation implies NCVE holds for the system. The $L_1$ norm is
also monotonically decreasing.

As a second example, we assume that the target rotates around the symmetric (third) axis, i.e., pure spin. The
initial condition of the target is $\theta^T = [\pi/20, \pi/10, 0]$, and $\omega^T = [0, 0, 2\pi/15]$, and the initial condition of the chaser
is $\theta^C = [0, 0, \pi/12]$, and $\omega^C = [0, 0.3\pi/25]$.

Simulation results with $Q = 1, R = 100$ are shown in Fig. 3. The Euler angles are given on the left side, and the
angler velocities on the right side, respectively. The first row is the motion of the target, the second row is that of the
chaser, the third row is the relative motion described by the nonlinear equation, and the fourth row is the relative
motion given by the linearized equation. We varied the value of $R$ and obtained the settling time, the $L_2$ and
the $L_1$ norm of the input torque as function of $R$ in Fig. 4. The settling time is given in the left figure, the $L_2$ norm of
the input torque in the middle, and the $L_1$ norm in the right figure, respectively.

We find that the settling time decreases as $R$ grows, since the second term of the cost function (17) gets dominant,
and also that the $L_2$ norm tends to zero. This confirms the NCVE property of the system. It is interesting to
note that the $L_1$ norm for the nonlinear equation is not monotonous decreasing with $R$, while the $L_1$ norm for the
linearized equation is monotonically decreasing. This result implies that an appropriate choice of $R$ minimizes the
$L_1$ norm of the input torque.
Fig. 3 motions for pure spin

Fig. 4 settling time, $L_2$ norm, and $L_1$ norm for $R$

5 Conclusion

We used Euler angle representation to describe the attitude of spacecrafts and stabilized the relative motion between a chaser satellite and a target satellite based on the linear-quadratic-regulator (LQR) theory. Our goal was to reduce the $L_1$ norm of the input torque, which is proportional to the amount of energy to control rotational motion of a spacecraft. Our simulation results show that the $L_1$ norm in monotonically decreasing has a minimum as a function of the penalty $R$. Thus the LQR theory is an effective method to design suboptimal controllers.

References