Numerical Approach for the Analysis of Flexible Beam motion with Time-Varying Length and Large Displacement Using Component Mode Synthesis

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Abstract: The floating frame of reference formulation (FFRF) is used in multibody dynamic analysis for flexible body motion with large displacement and rotation. The FFRF is extended to flexible body motion with time-varying length using a variable-domain beam element. In this approach, since the shape function is time-varying, the calculation cost increases.

In the present paper, the Craig-Bampton method, which is one of the mode synthesis methods, is applied to flexible body motion with time-varying length. This approach can reduce the calculation cost while maintaining the precision.

区分モード合成法を用いた大変位及び長さ変化を伴う柔軟梁の数値解析手法

摘要：大変位，大回転を伴う柔軟体の解析において一般的に使用されている FFR 法に，可変領域はり要素モデルを用いることで長さ変化を伴う柔軟体モデルに拡張された定式化を行う．この手法では，通常の FFR 法と比較すると長さ変化の影響で要素長さを含む形状関数が時変となり，結果的に計算コストが増える．

そこで本論文では，振動解析においてよく用いられるモード合成法の一つである Craig-Bampton 法を適用することで，計算精度を保持しつつ計算コストの低減化を行った．

1. INTRODUCTION

Studies on mechanical systems that incorporate large displacement and axially moving beams have received considerable attention in recent years. Such systems are in demand in a number of mechanical fields. Spacecraft antennas, elevators, and crane systems are applications that incorporate axially moving beams. In axially moving beam problems, slender beams may induce vibrations similar to those occurring in the spaghetti problem[1],[2].

For exact modeling of a flexible body, elastic beams consisting of multibody systems must be formulated as continuous systems. Furthermore, the displacement of the body and the rotation must be properly simulated for the formulation[3]. In the motion analysis of such a system, the deformation of flexible bodies should be described by a suitable model that takes into account large displacement, large rotation, and time-varying length[4]. In the normal of such systems, the deformation is not so large and the linear elastic theory can be applied in the numerical analysis. However, the calculation costs increase greatly due to the use of a finite element and time-dependent shape functions. Thus, in order to simplify such a problem, it is desirable to apply a calculation method to decrease the number of degrees of freedom and reduce the calculation cost while maintaining the precision.

In vibration analysis, the Craig-Bampton method[5], which is one of the component mode synthesis methods, is widely used to quickly perform an elastic response analysis while using the finite element method. In multibody dynamics, Gerstmayr and Ambrosio improved the calculation time by applying the Craig-Bampton method to reduce the number of degrees of freedom[6].

In the present study, the variable-domain beam model is formulated by combining the FFRF and the FEM, and applying the Craig-Bampton method to reduce the calculation cost while maintaining the accuracy. We then propose a high-speed approximation method that is useful for the analysis of the flexible body motion with large displacement, large rotation and time-varying length. Finally, the proposed approximate solution is compared to the conventional method through numerical examples, and we discuss the accuracy and computational efficiency of the proposed approach.

2. FORMULATION OF THE VARIABLE-DOMAIN BEAM ELEMENT MODEL USING FLOATING FRAME OF REFERENCE FORMULATION

In this chapter, for the formulation of flexible body motion with large displacement and time-varying length, the FFRF is applied to a variable-domain elements model, which is referred to as the variable finite element method on a floating frame of reference (VFE-FFR).

Consider the axially moving beam shown in Figure 1. The inertia frame and body frame are set with the rotation angles that describe the orientation of the body frame. The element coordinate system is attached as shown in Figure 1, and the displacement of a point along the neutral axis of the i-th element is given by the following expression:

$$X = L_i + x. \tag{1}$$

Solving for the element coordinate x, differentiating with respect to time, and using the relation $\dot{X} = \dot{L}$, we obtain

$$\dot{x} = \dot{L} - \dot{L}_i. \tag{2}$$

It is useful to introduce a new parameter d_i, as shown in Figure 1, which is defined as follow for each element:

$$d_i \equiv L - L_i = L\{1 - (i-1)/n\}. \tag{3}$$

Using Eq. (3), we can derive the differential operator with respect to time as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + d_i \frac{\partial}{\partial x} \tag{4}$$

2.1. ELEMENT CONFIGURATIONS

In the FFRF, variable-domain beam elements are described as shown in Figure 1.

The general displacement of any point on an elastic body is given as

$$\mathbf{r}^i(x,t) = \mathbf{R}(t) + \mathbf{A}\bar{\mathbf{u}}^i(x,t). \tag{5}$$

where $\mathbf{R}$ is the location of the origin of the body frame, $\mathbf{A}$ is the rotation matrix, and $\bar{\mathbf{u}}^i$ is the relative position vector of an arbitrary point on the flexible body in the body reference frame. This vector is described in terms of nodal coordinates $\mathbf{e}^i$ and the shape function $\mathbf{N}$, as follows:

$$\bar{\mathbf{u}}^i(x,t) = \mathbf{N}(x,t)\mathbf{e}^i(t). \tag{6}$$

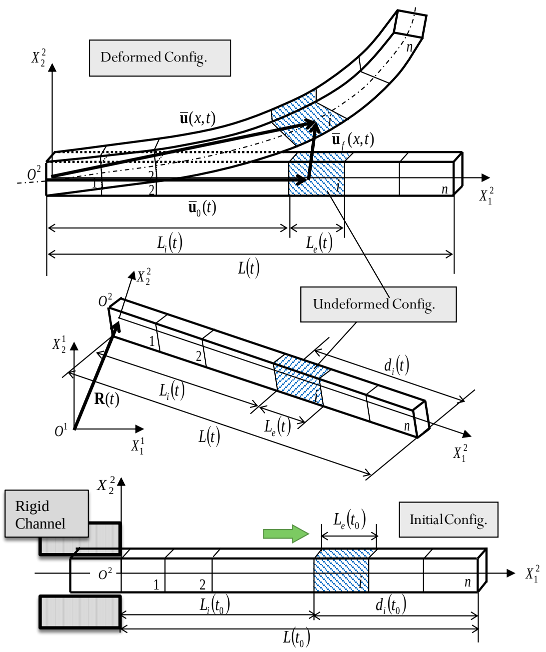


Figure 1 Element configuration on the FFR

2.2. VELOCITY AND ACCELERATION

Differentiating Eq (5) with respect to time using the differential operator given by Eq (4), the velocity and acceleration vectors are defined as follows:

$$\begin{aligned}\frac{d\mathbf{r}^i}{dt} &= \dot{\mathbf{R}} + \mathbf{A}_\theta \bar{\mathbf{u}}^i \dot{\theta} + \mathbf{A}\mathbf{N}\dot{\mathbf{e}}_f + \mathbf{A}\left(\dot{\mathbf{N}}\mathbf{e} + \mathbf{N}\dot{\mathbf{e}}_0\right) \\ &= \mathbf{L}^i \dot{\mathbf{q}}^i + \mathbf{A}\left(\dot{\mathbf{N}}\mathbf{e} + \mathbf{N}\dot{\mathbf{e}}_0\right)\end{aligned} \tag{7}$$

$$\frac{d^2\mathbf{r}^i}{dt^2} = \mathbf{L}^i \ddot{\mathbf{q}}^i + \mathbf{a}_d^i + \mathbf{a}_l^i \tag{8}$$

,

where $\mathbf{L}^i$ is a coefficient matrix, $\mathbf{L}^i = \begin{bmatrix}\mathbf{I} & \mathbf{A}_\theta \bar{\mathbf{u}}^i & \mathbf{A}\mathbf{N}\end{bmatrix}$, and the generalized coordinates are defined as $\mathbf{q}^i = \begin{bmatrix}\mathbf{R} & \theta & \mathbf{e}_f^i\end{bmatrix}^T$. In addition, $\mathbf{a}_d^i$ denotes the Coriolis acceleration due to the flexibility of the slender body, and $\mathbf{a}_l^i$ denotes the Coriolis acceleration caused by the length change velocity of the body, which is given as follows:

$$\mathbf{a}_d^i = \dot{\mathbf{L}}^i \dot{\mathbf{q}}^i, \quad \mathbf{a}_l^i = \frac{d\mathbf{v}_l^i}{dt}. \tag{9,10}$$

2.3. GOVERNING EQUATIONS

Using the principle of virtual work, the variational equations of motion of the flexible body element are given as follows:

$$\delta W_{extrenal}{}^i + \delta W_{inertia}{}^i + \delta W_{strain}{}^i = 0, \tag{11}$$

where the work by the generalized inertia force, the elastic force and the external force, respectively, are given as follows:

$$\begin{aligned}W_{inertia}{}^i &= -\int_{V^i} \rho \mathbf{r}^{iT} \frac{d^2\mathbf{r}^i}{dt^2} dV^i \\ &= -\delta \mathbf{q}^{iT}\left(\mathbf{M}^i \ddot{\mathbf{q}}^i - \mathbf{Q}_v{}^i - \mathbf{Q}_l{}^i\right)\end{aligned} \tag{12}$$

$$\begin{aligned}W_{strain}{}^i &= -\int_{V^i} \boldsymbol{\sigma}^{iT} \boldsymbol{\varepsilon}^i dV^i \\ &= -\delta \mathbf{q}^{iT} \mathbf{K}^i \mathbf{q}^i\end{aligned} \tag{13}$$

$$\begin{aligned}W_{external}{}^i &= \int_{V^i} \mathbf{r}^{iT} \mathbf{f}^i dV^i \\ &= \delta \mathbf{r}^{iT} \mathbf{F} = \delta \mathbf{q}^{iT} \mathbf{Q}_e{}^i.\end{aligned} \tag{14}$$

By Eq. (12), the generalized inertial force derives quadratic velocity vectors due to the elastic deformation and changes in length.

$$\mathbf{M}^i = \int_{V^i} \rho \mathbf{L}^{iT} \mathbf{L}^i dV^i = \begin{bmatrix}\mathbf{M}_{RR} & \mathbf{M}_{R\theta} & \mathbf{M}_{Rf} \\ & \mathbf{M}_{\theta\theta} & \mathbf{M}_{\theta f} \\ sym. & & \mathbf{M}_{ff}\end{bmatrix} \tag{15}$$

$$\mathbf{Q}_v{}^i = -\int_{V^i} \rho \mathbf{L}^{iT} \mathbf{a}_d{}^i dV^i = \begin{bmatrix}\left(\mathbf{Q}_v^i\right)_R \\ \left(\mathbf{Q}_v^i\right)_\theta \\ \left(\mathbf{Q}_v^i\right)_f\end{bmatrix} \tag{16}$$

$$\mathbf{Q}_l{}^i = -\int_{V^i} \rho \mathbf{L}^{iT} \mathbf{a}_l{}^i dV^i = \begin{bmatrix}\left(\mathbf{Q}_l^i\right)_R \\ \left(\mathbf{Q}_l^i\right)_\theta \\ \left(\mathbf{Q}_l^i\right)_f\end{bmatrix} \tag{17}$$

Then, by substituting Eqs. (12) through (14) into Eq. (11), the governing equations of the system are derived as

$$\begin{gathered}\begin{bmatrix}\mathbf{M}_{RR} & \mathbf{M}_{R\theta} & \mathbf{M}_{Rf} \\ & M_{\theta\theta} & \mathbf{M}_{\theta f} \\ sym. & & \mathbf{M}_{ff}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{R}} \\ \ddot{\theta} \\ \ddot{\mathbf{e}}_f\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ & 0 & \mathbf{0} \\ sym. & & \mathbf{K}_{ff}\end{bmatrix}\begin{bmatrix}\mathbf{R} \\ \theta \\ \mathbf{e}_f\end{bmatrix} \\ - \begin{bmatrix}\left(\mathbf{Q}_v^i\right)_R \\ \left(Q_v^i\right)_\theta \\ \left(\mathbf{Q}_v^i\right)_f\end{bmatrix} - \begin{bmatrix}\left(\mathbf{Q}_l^i\right)_R \\ \left(Q_l^i\right)_\theta \\ \left(\mathbf{Q}_l^i\right)_f\end{bmatrix} - \begin{bmatrix}\left(\mathbf{Q}_e^i\right)_R \\ \left(Q_e^i\right)_\theta \\ \left(\mathbf{Q}_e^i\right)_f\end{bmatrix} + \mathbf{C}_\mathbf{q}^T \boldsymbol{\lambda} = \mathbf{0}\end{gathered} \tag{18}$$

,

where $\mathbf{C}_\mathbf{q}^T \boldsymbol{\lambda}$ is the force to break the constraints, which is derived from the constraint equations.

3. COMPONENT MODE SYNTHESIS METHOD

In flexible multibody dynamics, the component mode synthesis method can be used to reduce the number of equations of complex objectives, by taking into account only the mode with sufficient flexibility. In the FFRF, the governing equations representing the flexible portion of the entire system are derived from the third row of Eq. (18), as follows:

$$\mathbf{M}_{ff} \ddot{\mathbf{e}}_f + \mathbf{K}_{ff} \mathbf{e}_f = \mathbf{F}_f \tag{19}$$

where $\mathbf{F}_f$ contains forces, terms depending on the rigid body translation and rotation variables, and the corresponding quadratic velocity vector, which is related to $\mathbf{e}_f$ and $\dot{\mathbf{e}}_f$. However, the modal analysis is based only on the left-hand side of Eq. (19).

The elastic coordinates are split into a boundary and internal coordinates, as follows: $\mathbf{e}_f^{'} = \begin{bmatrix}\mathbf{e}_b \\ \mathbf{e}_i\end{bmatrix}$, and the mass and stiffness matrices are also partitioned accordingly as

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{e}}_b \\ \ddot{\mathbf{e}}_i \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{e}_b \\ \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \mathbf{F}_b \\ \mathbf{F}_i \end{bmatrix}, \tag{20}$$

where subscript b refers to the boundary region defined by the joint constraint, and subscript i refers to the interior region. In order to represent the nodal coordinates with respect to the interior region using mode coordinates, two new matrices are introduced, namely, the constraint mode matrix derived from Guyan reduction, as shown in Eq. (21), and the normal mode matrix that can be obtained by solving the eigen-equation, as shown in Eq. (22),

$$\mathbf{\Phi}_c \equiv -\mathbf{K}_{bb}{}^{-1}\mathbf{K}_{ib} \tag{21}$$

$$\left(\mathbf{K}_{ii} - \Omega_k^2 \mathbf{M}_{ii}\right)\mathbf{\Phi}_{nk} = \mathbf{0} \tag{22}$$

where Ω_k and $\mathbf{\Phi}_{nk}$ in Eq. (22) are the k-th natural angular frequency and its modal vector, respectively. In the Craig-Bampton method, using Eqs. (21) and (22), nodal coordinates $\hat{\mathbf{e}}$ can be represented as follows:

$$\hat{\mathbf{e}} = \mathbf{\Phi}_{CB}\overline{\mathbf{q}} \tag{23}$$

$$\mathbf{\Phi}_{CB} \equiv \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Phi}_c & \mathbf{\Phi}_n \end{bmatrix} \tag{24}$$

where $\mathbf{\Phi}_{CB}$ is the transformation matrix, $\overline{\mathbf{q}} = \begin{bmatrix} \mathbf{e}_b^T & \boldsymbol{\xi}^T \end{bmatrix}^T$, and $\boldsymbol{\xi}$ is the mode coordinates.

In order to apply the Craig-Bampton method to the FFRF, as discussed in the preceding chapter, the extended transformation matrix is introduced as follows:

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{CB} \end{bmatrix}. \tag{25}$$

Using Eq. (26), the final generalized coordinates can be determined by the following equation:

$$\begin{bmatrix} \mathbf{R} \\ \theta \\ \hat{\mathbf{e}} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{R} \\ \theta \\ \overline{\mathbf{q}} \end{bmatrix} \tag{26}$$

where $\overline{\mathbf{M}}$, $\overline{\mathbf{K}}$, $\overline{\mathbf{Q}}_v$, and $\overline{\mathbf{Q}}_l$ are the transformed mass matrix, the stiffness matrix, and the quadratic velocity vectors due to the elastic deformation and changes in length, respectively, as defined by the following equations:

$$\overline{\mathbf{M}} = \mathbf{T}^T\mathbf{M}\mathbf{T} = \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{R\theta} & \mathbf{M}_{Rf}\mathbf{\Phi}_{CB} \\ & M_{\theta\theta} & \mathbf{M}_{\theta f}\mathbf{\Phi}_{CB} \\ sym. & & \mathbf{\Phi}_{CB}{}^T\mathbf{M}_{ff}\mathbf{\Phi}_{CB} \end{bmatrix} \tag{27}$$

$$\overline{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & 0 & \mathbf{0} \\ sym. & & \mathbf{\Phi}_{CB}{}^T\mathbf{K}_{ff}\mathbf{\Phi}_{CB} \end{bmatrix} \tag{28}$$

$$\overline{\mathbf{Q}}_v = \mathbf{T}^T\mathbf{Q}_v = \begin{bmatrix} \left(\mathbf{Q}_v^i\right)_R \\ \left(Q_v^i\right)_\theta \\ \mathbf{\Phi}_{CB}{}^T\left(\mathbf{Q}_v^i\right)_f \end{bmatrix} \tag{29}$$

$$\overline{\mathbf{Q}}_l = \mathbf{T}^T\mathbf{Q}_l = \begin{bmatrix} \left(\mathbf{Q}_l^i\right)_R \\ \left(Q_l^i\right)_\theta \\ \mathbf{\Phi}_{CB}{}^T\left(\mathbf{Q}_l^i\right)_f \end{bmatrix}. \tag{30}$$

4. NUMERICAL EXAMPLE AND COMPARISON OF THE PROPOSED METHODS

The planar motion of an elastic pendulum system representing a flexible, slender body with large rotation and time-varying length has been analyzed and a comparison between the proposed analysis method and the VFE-FFR has been carried out. The illustrated examples are the well known spaghetti and reverse spaghetti problems. Figure 2 shows these analysis models of elastic pendulum with time varying length in extension and extraction. The initial angle of the pendulum is $\theta(t=0) = -5\pi/12$, the number of elements is $N=5$, and the material parameters are listed in Table 1. In order to remove a redundant rigid body mode, the reference condition of the cantilever beam is imposed. When the pendulum is extended, the velocity of the length change of the entire beam is V= 0.1 [*m/s*], and when the pendulum is extracted, the velocity of the length change of the entire beam is V= -0.1 [*m/s*]. The constraints of the pin-joint at the upper point are given by the following equations:

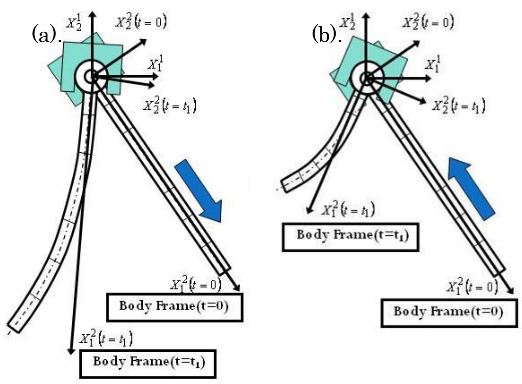
$$\mathbf{C}(\mathbf{q}) = \mathbf{R} = \mathbf{0} \tag{31}$$

Figure 3 shows the numerical results of the shape of the beam extension and extraction using the proposed method. Figure 4 shows the absolute position of the pendulum tip point on the *X*-axis and the rotation of the body frame from the inertial frame.

When the beam extends, the results for both methods correspond well, as shown in Figure 4(a). However, as shown in Figure 4(b), when the beam extracts, the error of the CMS-VFE-FFRF against the VFE-FFRF increases with time. Because of the influence of higher vibration modes in the elastic pendulum, the error increases as the beam extracts.

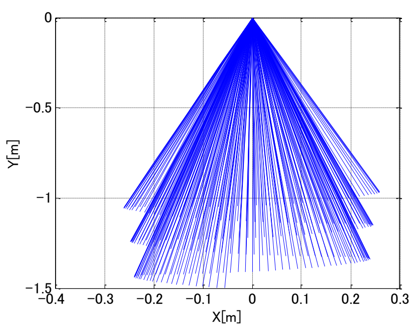
Table 1 Material parameters

Property	Quantity
Intial length [m]	1
Cross section area [m^2]	π×0.03×0.03
Material density [kg/m^3]	2000
Modulus of elasticity[Pa]	2.00E+08

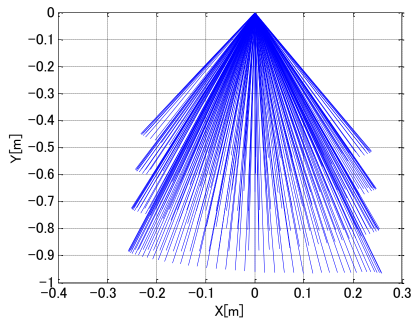


(a) Extension, (b) Extraction

Figure 2 Model of the elastic pendulum

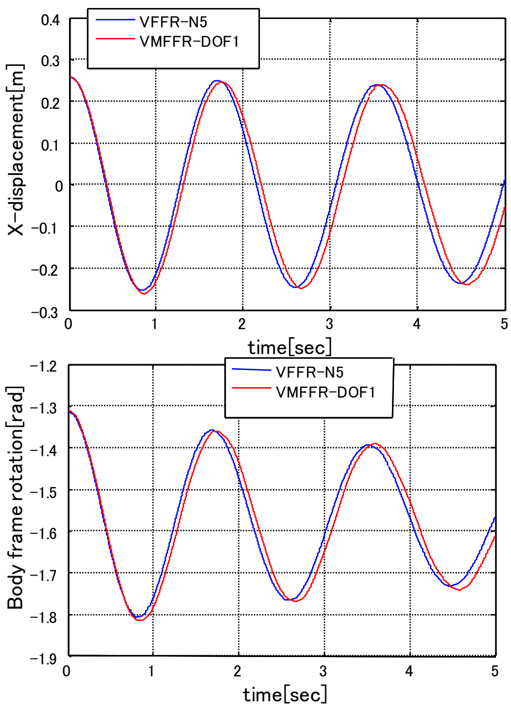


(a). Extension

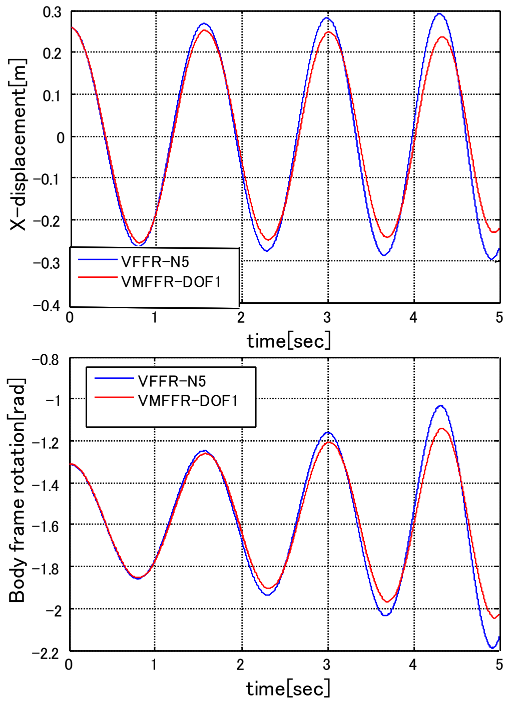


(b). Extraction

Figure 3 Shape of the neutral axis of the beam



(a).Extension



(b). Extraction

Figure 4 X-displacement of the pendulum tip in the rotation of the body frame

Table 2 Calculation time

Calculation Time		Percentage
【Ex.1】extended length		
VFE-FFRF(N=5)	2014.7311[sec]	—
CMS-VFE-FFRF(DOF.1)	1437.9909[sec]	71.37[%]
【Ex.2】shortened length		
VFE-FFRF(N=5)	2031.1189[sec]	—
CMS-VFE-FFRF(DOF.1)	1418.9496[sec]	69.86[%]

The calculation times obtained using a 2.93GHz Intel(R) Xeon(R) are listed Table 2. In both examples, the calculation cost using the proposed method can be reduced by approximately 30 [%], as compared to that using the VFE-FFR. Thus, the proposed approximate method is effective for the case in case in which the analyzed system dominated by lower vibration modes.

5. SUMMARY AND CONCLUSIONS

In the present study, for the purpose of proposing an efficient method for analysis of flexible beam motion with large displacement and time-varying length, we apply classification mode synthesis to the VFE-FFR. This method can reduce the matrix size of shape integrals required for the FFRF by applying the Craig-Bampton method to the model composed of variable-domain elements. The proposed method reduced the computational cost of motion analysis by approximately 70 [%] for both the extension and extraction of the pendulum. The proposed approximate method was demonstrated to effectively simulate the motion of a system that dominated by lower vibration modes while maintaining a certain level of accuracy.

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