A Study on Singular States of CMGs

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When four single-gimbal control moment gyros are used for attitude control of spacecraft, there are singular states where gimbal rates cannot be determined from attitude control torques. The characteristics of singular states and a method to avoid the singularity are considered in this study.

CMG の特異点に関する考察

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本稿では 4 台のピラミッド型配置 CMG を用いた特異点について考察し，CMG の微小ジンバル角の 2 次項まで用いたときの特異点の通過可能性について新たな定義付けを行い，その定義を用いた特異点の通過可能性について検討する．さらに，CMG の微小ジンバル角の 2 次項まで用いるときの，ジンバル角の解を求める方法について検討し，そのときに得られた微小ジンバル角の解が特異点における CMG の制御に対して有効であることを示す.

1 Introduction

Maneuver agility in attitude control of spacecraft has been highly required in recent years. Control moment gyros (CMG) have attracted much attention as attitude control actuators in the agile attitude control[1]–[5]. In the case of single-gimbal CMGs, a pyramid-type arrangement of four CMGs is the most general one. Although this type of CMG arrangement utilizes the angular momentum of CMGs effectively, a singular state of CMGs where desirable control torques cannot be realized sometimes occurs corresponding to gimbal angles of CMGs. In the singular state, the ideal gimbal rates cannot be obtained. However, when the relation between the small change of the angular momentum of CMGs and that of the gimbal angles is focused, the change of the gimbal angles can be calculated with the consideration of second-order terms of the small quantities. In this paper, the method to continue the attitude control in the singular state of the pyramid type of CMGs is considered by taking the second-order terms of the small changes of the gimbal angles into account. The method to obtain exactly the small changes of the gimbal angles is proposed here. Because plural solutions of the gimbal angles generally exist, the method to obtain numerically the solution that minimizes the sum of the square of the gimbal angle changes is also considered. Furthermore, results of the attitude control of spacecraft by using the obtained solution are shown in the paper.

2 Pyramid-type Arrangement of CMGs

2.1 Attitude control torque

Four single-gimbal CMGs that are placed in a pyramid-type arrangement shown in Fig. 1 are focused in this paper. Let a unit vector in the direction of the gimbal axis of the ith CMG be $\hat{\boldsymbol{g}}_i (i = 1, 2, 3, 4)$. These unit vectors are expressed in the body-fixed coordinates as

$$\hat{\boldsymbol{g}}_1 = \begin{bmatrix} \sin\beta \\ 0 \\ \cos\beta \end{bmatrix}, \quad \hat{\boldsymbol{g}}_2 = \begin{bmatrix} 0 \\ \sin\beta \\ \cos\beta \end{bmatrix},$$
$$\hat{\boldsymbol{g}}_3 = \begin{bmatrix} -\sin\beta \\ 0 \\ \cos\beta \end{bmatrix}, \quad \hat{\boldsymbol{g}}_4 = \begin{bmatrix} 0 \\ -\sin\beta \\ \cos\beta \end{bmatrix} \tag{1}$$

where β is an inclination angle of the pyramid. Let a rotation angle of the gimbal of the ith CMG be θ_i. Then, the unit vector in the direction of the angular momentum of the ith CMG, $\hat{\boldsymbol{h}}_i$, is expressed as follows where the angular momentum of the ith CMG at $\theta_i = 0$ lies in the xy-plane in the body-fixed frame:

$$\hat{\boldsymbol{h}}_1 = \begin{bmatrix} -\sin\theta_1\cos\beta \\ \cos\theta_1 \\ \sin\theta_1\sin\beta \end{bmatrix}, \quad \hat{\boldsymbol{h}}_2 = \begin{bmatrix} -\cos\theta_2 \\ -\sin\theta_2\cos\beta \\ \sin\theta_2\sin\beta \end{bmatrix},$$
$$\hat{\boldsymbol{h}}_3 = \begin{bmatrix} \sin\theta_3\cos\beta \\ -\cos\theta_3 \\ \sin\theta_3\sin\beta \end{bmatrix}, \quad \hat{\boldsymbol{h}}_4 = \begin{bmatrix} \cos\theta_4 \\ \sin\theta_4\cos\beta \\ \sin\theta_4\sin\beta \end{bmatrix} \tag{2}$$

If the magnitude of the angular momentum of each CMG is defined as h_w, the whole angular momentum of the CMGs, $\boldsymbol{h}$, is expressed as

$$\boldsymbol{h} = \sum_i h_w \hat{\boldsymbol{h}}_i \tag{3}$$

The torque $\boldsymbol{\tau}$ that is applied to the CMGs is expressed by the following equation:

$$\boldsymbol{\tau} = \sum_i h_w \dot{\hat{\boldsymbol{h}}}_i = h_w \sum_i \frac{\partial \hat{\boldsymbol{h}}_i}{\partial \theta_i}\dot{\theta}_i = h_w \boldsymbol{A}\dot{\boldsymbol{\theta}} \tag{4}$$

where the attitude control torque of the spacecraft, $\boldsymbol{\tau_c}$, becomes $\boldsymbol{\tau}_c = -\boldsymbol{\tau}$. Let $\hat{\boldsymbol{\tau}}_i$ be defined by

$$\hat{\boldsymbol{\tau}}_i = \frac{\partial \hat{\boldsymbol{h}}_i}{\partial \theta_i}, \tag{5}$$

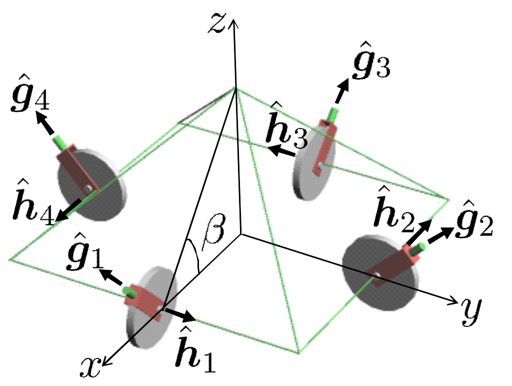


Fig 1: Pyramid-type arrangement ($\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$)

$\boldsymbol{A}$ and $\boldsymbol{\theta}$ are expressed as follows:

$$\boldsymbol{A} = \begin{bmatrix} \hat{\boldsymbol{\tau}}_1 & \hat{\boldsymbol{\tau}}_2 & \hat{\boldsymbol{\tau}}_3 & \hat{\boldsymbol{\tau}}_4 \end{bmatrix}$$
$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^T$$

2.2 Singular state of CMGs

The singular state of CMGs means the state where the rank of matrix $\boldsymbol{A}$ becomes less or equal 2. In this singular state, an arbitrary attitude control torque $\boldsymbol{\tau}_c$ cannot be realized.

In the singular state, there is a certain unit vector $\hat{\boldsymbol{u}}$ that is orthogonal with each column of matrix $\boldsymbol{A}$, because the rank of matrix $\boldsymbol{A}$ is deficient. This vector is called a singular vector. The singular vector is obtained by the singular-value decomposition of matrix $\boldsymbol{A}$ as described later. In general, the singular states of the pyramid-type arrangement of CMGs are classified into the following three cases by the sign of the inner product $\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{h}}_i$: 4H where all inner products have the same signs, 2H where one of the inner products has the different sign with the others, and 0H where two of the inner products have + signs and two of the inner products have − signs. All of the 4H singular states and a part of the 2H singular states compose the angular momentum envelope of the CMGs.

3 Passability of Singular States

3.1 Definition of passability

In the case of the singular states of CMGs, Eq. (4) has no solution with $\dot{\boldsymbol{\theta}}$, because matrix $\boldsymbol{A}$ is rank-deficient. Equation (4) expresses the relation between the attitude control torque and the gimbal rates at a certain time instant, and this equation can be regarded as the relation between the change of the angular momentum and that of the gimbal angles when the time is expanded to a finite value. When the angular momentum of CMGs is expanded around a certain gimbal angles $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, the change of the angular momentum $\Delta\boldsymbol{h} = \boldsymbol{h} - \boldsymbol{h}_0$ are expanded with the change of the gimbal angles $\Delta\boldsymbol{\theta} = \boldsymbol{\theta} - \boldsymbol{\theta}_0$ as

$$\Delta\boldsymbol{h} = \left(\left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right) \Delta\boldsymbol{\theta} + \frac{1}{2}\Delta\boldsymbol{\theta}^T \left(\left.\frac{\partial^2 \boldsymbol{h}}{\partial \boldsymbol{\theta}^2}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right) \Delta\boldsymbol{\theta} + \cdots \quad (6)$$

where $\boldsymbol{h}_0$ is the angular momentum of the CMGs at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. In the above equation, $\partial\boldsymbol{h}/\partial\boldsymbol{\theta}$ means matrix $\boldsymbol{A}$, and $\partial^2\boldsymbol{h}/\partial\boldsymbol{\theta}^2$ is a particular matrix whose (i, j) component becomes a vector $\partial^2\boldsymbol{h}/\partial\theta_i\partial\theta_j$. Let us express this equation up to the second-order terms of $\Delta\boldsymbol{\theta}$ as

$$\Delta\boldsymbol{h} = h_w \left(\boldsymbol{A}\Delta\boldsymbol{\theta} + \frac{1}{2}\Delta\boldsymbol{\theta}^T \boldsymbol{H} \Delta\boldsymbol{\theta} \right) \quad (7)$$

In this equation, the (i, j) component of matrix $\boldsymbol{H}$, $\boldsymbol{H}_{ij}$, is a 3-dimensional vector, and this is expressed as follows because $\hat{\boldsymbol{h}}_i$ depends only on θ_i:

$$\boldsymbol{H}_{ij} = \sum_k \frac{\partial^2 \hat{\boldsymbol{h}}_k}{\partial\theta_i \partial\theta_j} = \begin{cases} -\hat{\boldsymbol{h}}_i & (i = j) \\ \boldsymbol{0} & (i \neq j) \end{cases} \quad (8)$$

Based on Eq. (7), the gimbal rates $\dot{\boldsymbol{\theta}}$ can be obtained approximately in singular states of CMGs. In order to obtain $\Delta\boldsymbol{\theta}$ that satisfies Eq. (7) in the singular states, matrix $\boldsymbol{A}$ is expressed by the singular-value decomposition as

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \sum_i \sigma_i \hat{\boldsymbol{u}}_i \hat{\boldsymbol{v}}_i^T \quad (9)$$

where σ_i is the singular value of matrix $\boldsymbol{A}$, and $\boldsymbol{U}$ and $\boldsymbol{V}$ are a 3×3 orthogonal matrix and a 4×4 orthogonal matrix, respectively. Then, $\hat{\boldsymbol{u}}_i$ and $\hat{\boldsymbol{v}}_i$ are the ith column vector of $\boldsymbol{U}$ and that of $\boldsymbol{V}$, respectively. When matrix $\boldsymbol{A}$ becomes singular, it follows in general that $\sigma_1 \geq \sigma_2 > 0$, $\sigma_3 = 0$. Therefore, the range space of matrix $\boldsymbol{A}$ is spanned by vectors $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$, whereas $\hat{\boldsymbol{u}}_3$ becomes the preceding singular vector $\hat{\boldsymbol{u}}$. Let $\Delta\boldsymbol{\theta}$ be transformed as

$$\Delta\boldsymbol{\theta} = \boldsymbol{V}\boldsymbol{\alpha} \quad (10)$$

By substituting the above relation into Eq. (7), the following equation is obtained:

$$\Delta\boldsymbol{h} = h_w \left(\boldsymbol{A}\boldsymbol{V}\boldsymbol{\alpha} + \frac{1}{2}\boldsymbol{\alpha}^T \boldsymbol{G} \boldsymbol{\alpha} \right), \quad \boldsymbol{G} = \boldsymbol{V}^T \boldsymbol{H} \boldsymbol{V} \quad (11)$$

where $\boldsymbol{G}$ is a 4×4 matrix whose component is a 3-dimensional vector. By taking inner products of the above equation with $\hat{\boldsymbol{u}}_1$, $\hat{\boldsymbol{u}}_2$, and $\hat{\boldsymbol{u}}_3$, the following relations are obtained:

$$\Delta h_1 = \Delta\boldsymbol{h} \cdot \hat{\boldsymbol{u}}_1 = h_w \left(\sigma_1 \alpha_1 + \frac{1}{2}\boldsymbol{\alpha}^T \boldsymbol{G}_1 \boldsymbol{\alpha} \right) \quad (12)$$

$$\Delta h_2 = \Delta\boldsymbol{h} \cdot \hat{\boldsymbol{u}}_2 = h_w \left(\sigma_2 \alpha_2 + \frac{1}{2}\boldsymbol{\alpha}^T \boldsymbol{G}_2 \boldsymbol{\alpha} \right) \quad (13)$$

$$\Delta h_3 = \Delta\boldsymbol{h} \cdot \hat{\boldsymbol{u}}_3 = \frac{1}{2} h_w \boldsymbol{\alpha}^T \boldsymbol{G}_3 \boldsymbol{\alpha} \quad (14)$$

where $\boldsymbol{G}_i$ is a 4×4 matrix that is obtained from the inner product of each component of $\boldsymbol{G}$ with $\hat{\boldsymbol{u}}_i$. Because σ_1 and σ_2 take positive values even when the CMGs are in the singular state, the first-order solutions of α_1 and α_2 are obtained from Eqs. (12) and (13), respectively.

The following two definitions of the passability of CMGs are considered here:

- Definition 1
Equation (14) has always real solutions with α_3 and α_4 when α_1 and α_2 from Eqs. (12) and (13) are substituted into Eq. (14).

- Definition 2
Equation (7) has a real successive solution with $\Delta\boldsymbol{\theta}$ in the vicinity of $|\Delta\boldsymbol{h}| = 0$.

Definition 1 is the definition that has been used[3]. This definition means that the angular momentum component in the direction of vector $\hat{\boldsymbol{u}}$ can be generated by the null motion that is spanned by the set of vectors $\hat{\boldsymbol{v}}_3$ and $\hat{\boldsymbol{v}}_4$[4]. The singular state becomes passable when the eigenvalues of 2×2 partial matrix of $\boldsymbol{G}_3$ that corresponds to α_3 and α_4 have different signs.

Definition 2 means that the very small real solution of $\Delta\boldsymbol{\theta}$ is obtained when the very small $\Delta\boldsymbol{h}$ is given. $\Delta\boldsymbol{\theta} = \boldsymbol{0}$ clearly becomes the solution for $\Delta\boldsymbol{h} = \boldsymbol{0}$. Then, if the very small real $\Delta\boldsymbol{\theta}$ exists for the very small $\Delta\boldsymbol{h}$, the singular state is regarded as passable in the direction of $\Delta\boldsymbol{h}$. Because definition 2 is based on the solution of Eq. (7), it is more practical to obtain $\Delta\boldsymbol{\theta}$. However, definition 2 is different from definition 1, because it depends on the direction of $\Delta\boldsymbol{h}$.

3.2 Comparison of Definitions 1 and 2

3.2.1 Arbitrary direction of $\Delta\boldsymbol{h}$

Whether the singular state is passable or not by definition 2 depends on the direction of $\Delta\boldsymbol{h}$. Let a unit vector in the direction of $\Delta\boldsymbol{h}$ be $\hat{\boldsymbol{p}}$, and $\Delta\boldsymbol{h}$ is expressed as $\Delta\boldsymbol{h} = \delta\hat{\boldsymbol{p}}$ where δ is a small positive value. Then, Eq. (14) is expressed as follows:

$$\delta\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{u}}_3 = \frac{h_w}{2}\hat{\boldsymbol{\alpha}}^T \boldsymbol{G}_3 \hat{\boldsymbol{\alpha}} \tag{15}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{G}_3$ are decomposed as follows corresponding to two components, α_1 and α_2, and two components, α_3 and α_4:

$$\boldsymbol{\alpha}_1 = \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}, \quad \boldsymbol{\alpha}_2 = \begin{bmatrix}\alpha_3\\ \alpha_4\end{bmatrix}, \quad \boldsymbol{G}_3 = \begin{bmatrix}\boldsymbol{G}_{311} & \boldsymbol{G}_{312}\\ \boldsymbol{G}_{312}^T & \boldsymbol{G}_{322}\end{bmatrix}$$

Then, Eq. (15) is expressed as

$$\frac{h_w}{2}\boldsymbol{\alpha}_2^T \boldsymbol{G}_{322}\boldsymbol{\alpha}_2 + h_w \boldsymbol{\alpha}_1^T \boldsymbol{G}_{312}\boldsymbol{\alpha}_2 + \frac{h_w}{2}\boldsymbol{\alpha}_1^T \boldsymbol{G}_{311}\boldsymbol{\alpha}_1 = \delta\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{u}}_3 \tag{16}$$

Furthermore, $\boldsymbol{G}_{322}$ is diagonalized by using 2×2 orthogonal matrix $\boldsymbol{P}$ as

$$\boldsymbol{G}_{322} = \boldsymbol{P}^T \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \boldsymbol{P} \tag{17}$$

where λ_1 and λ_2 are real eigenvalues of $\boldsymbol{G}_{322}$. Then, $\boldsymbol{\alpha}_2$ is transformed into ξ_1 and ξ_2 as

$$\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \boldsymbol{P}\boldsymbol{\alpha}_2$$

Equation (16) is expressed as follows:

$$\lambda_1\xi_1^2 + \lambda_2\xi_2^2 + c_1\xi_1 + c_2\xi_2 = c_0 \tag{18}$$

where

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = 2\boldsymbol{P}\boldsymbol{G}_{312}^T\boldsymbol{\alpha}_1, \quad c_0 = \frac{2\delta}{h_w}\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{u}}_3 - \boldsymbol{\alpha}_1^T \boldsymbol{G}_{311}\boldsymbol{\alpha}_1$$

If the sign of c_0 is the same with that of λ_1 or λ_2, Eq. (18) has a real solution for ξ_1 and ξ_2. Assuming that variables c_0, α_1, and α_2 are the first-order small quantities, the sign of c_0 is almost determined by that of $\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{u}}_3$. Therefore, if either the sign of λ_1 or λ_2 is the same with that of $\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{u}}_3$, ξ_1 and ξ_2 have real solutions. Because the singular state becomes passable in this condition, this is the passability condition by definition 2 for the direction $\hat{\boldsymbol{p}}$. The passability condition by definition 1 is that the sign of λ_1 is different from that of λ_2. This means that the singular state is passable in any directions by definition 2.

3.2.2 Direction of singular vector $\boldsymbol{u}_3$

Let us consider here the passability of the singular state when $\Delta\boldsymbol{h}$ is in the direction of singular vector $\hat{\boldsymbol{u}}_3$. By substituting $\Delta\boldsymbol{h} = \delta\hat{\boldsymbol{u}}_3$ into Eq. (7) and taking inner products with singular vector $\hat{\boldsymbol{u}}_3$, the following equation is obtained because $\hat{\boldsymbol{u}}_3$ is orthogonal with each column of matrix $\boldsymbol{A}$:

$$\delta = -\frac{h_w}{2}\sum_i \left(\hat{\boldsymbol{u}}_3 \cdot \hat{\boldsymbol{h}}_i\right)\Delta\theta_i^2 \tag{19}$$

In the singular state, the change of the angular momentum of CMGs in the direction of the singular vector is expressed in the above equation. Because the signs of $\hat{\boldsymbol{u}}_3 \cdot \hat{\boldsymbol{h}}_i$ are the same in the 4H singular states, the change of the angular momentum in the direction of $\delta > 0$ is impossible. This is the reason why 4H singular states compose the momentum envelope. Therefore, 4H singular states are impassable by definition 1, whereas they are impassable in the direction of $\delta > 0$ by definition 2.

3.3 Computation method of solution at singular states

3.3.1 Solution of simultaneous second-order equations

Although the passability of singular states by definition 2 can be checked as described above, simultaneous second-order equations (7) need to be solved to obtain the values of $\Delta\boldsymbol{\theta}$ that realize $\Delta\boldsymbol{h}$ at the singular state. The equations are expressed in the following form:

$$\Delta\boldsymbol{h} = h_w\boldsymbol{A}\begin{bmatrix}\Delta\theta_1\\ \Delta\theta_2\\ \Delta\theta_3\\ \Delta\theta_4\end{bmatrix} - \frac{h_w}{2}\boldsymbol{B}\begin{bmatrix}\Delta\theta_1^2\\ \Delta\theta_2^2\\ \Delta\theta_3^2\\ \Delta\theta_4^2\end{bmatrix} \tag{20}$$

$$\boldsymbol{B} = \begin{bmatrix}\hat{\boldsymbol{h}}_1 & \hat{\boldsymbol{h}}_2 & \hat{\boldsymbol{h}}_3 & \hat{\boldsymbol{h}}_4\end{bmatrix}$$

Because components of matrix $\boldsymbol{H}$ are zero vectors except for the diagonal components, $\Delta\theta_i^2$ terms only appear as the second-order terms in the above equation.

These equations consist of three equations for four variables in $\Delta\boldsymbol{\theta}$. Therefore, one variable in $\Delta\boldsymbol{\theta}$ can be set arbitrarily. By setting this arbitrary variable as $\Delta\theta_4$, these equations are expressed as follows:

$$\begin{aligned}&\Delta\boldsymbol{h} - h_w\hat{\boldsymbol{\tau}}_4\Delta\theta_4 - \frac{h_w}{2}\hat{\boldsymbol{h}}_4\Delta\theta_4^2\\ &= h_w\boldsymbol{A}_4\begin{bmatrix}\Delta\theta_1\\ \Delta\theta_2\\ \Delta\theta_3\end{bmatrix} - \frac{h_w}{2}\boldsymbol{B}_4\begin{bmatrix}\Delta\theta_1^2\\ \Delta\theta_2^2\\ \Delta\theta_3^2\end{bmatrix}\end{aligned} \tag{21}$$

where $\boldsymbol{A}_4$ and $\boldsymbol{B}_4$ express 3×3 matrices that consist of the first, the second, and the third column of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. From these equations, $\Delta\theta_1^2$, $\Delta\theta_2^2$, and $\Delta\theta_3^2$ are expressed as linear equations with respect to $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$. By using this characteristics, simultaneous second-order equations with respect to $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$ can be solved by using the Gröbner bases. These bases are expressed as follows:

$$\begin{aligned}g_3(\Delta\theta_3, \Delta\theta_4) &= 0\\ g_2(\Delta\theta_2, \Delta\theta_3, \Delta\theta_4) &= 0\\ g_1(\Delta\theta_1, \Delta\theta_3, \Delta\theta_4) &= 0\end{aligned} \tag{22}$$

In the above equations, $g_3(\Delta\theta_3, \Delta\theta_4) = 0$ becomes an eighth-order algebraic equation with respect to $\Delta\theta_3$. From this equation, the value of $\Delta\theta_3$ can be obtained when the value of arbitrary variable $\Delta\theta_4$ is set. Two equations, $g_2(\Delta\theta_2, \Delta\theta_3, \Delta\theta_4) = 0$ and $g_1(\Delta\theta_1, \Delta\theta_3, \Delta\theta_4) = 0$, are linear equations with respect to $\Delta\theta_2$ and $\Delta\theta_1$, respectively. When the value of $\Delta\theta_3$ is obtained, the values of $\Delta\theta_2$ and $\Delta\theta_1$ are easily obtained by solving these equations. The merit of obtaining solutions by the Gröbner bases is that all the solutions can be obtained by solving the eighth-order algebraic equation, $g_3(\Delta\theta_3, \Delta\theta_4) = 0$. Therefore, the solution that minimizes the norm of $\Delta\boldsymbol{\theta}$ among the solutions is also obtained.

3.3.2 Arbitrary parameter

As described above, one of $\Delta\theta_1 \sim \Delta\theta_4$ should be set as an arbitrary parameter in order to obtain the solution of $\Delta\boldsymbol{\theta}$. The norm of the solution differs depending on the value of the arbitrary parameter. In order to obtain the solution with the minimum norm, the value of the parameter is desirable to be close to 0. Therefore, the arbitrary parameter among $\Delta\theta_1 \sim \Delta\theta_4$ is selected as follows: the parameter is chosen from $\Delta\theta_1$ to $\Delta\theta_4$ one by one, the solution is obtained by setting the value of the arbitrary parameter at 0. The parameter is determined from $\Delta\theta_1 \sim \Delta\theta_4$ so that the minimum norm of the solution is obtained.

3.3.3 Determination of parameter value by Newton method

When the arbitrary parameter is selected from $\Delta\theta_1$ to $\Delta\theta_4$ as described above, the next problem is how to determine the parameter value. The norm of $\Delta\boldsymbol{\theta}$ differs depending on the parameter value. In order to set the parameter value, the optimum value is to be searched by numerical methods such as the Newton method or the bisection method.

In the case of the Gröbner bases, the derivatives needed for the Newton method can also be obtained by the bases. Therefore, the Newton method is focused hereafter. Firstly, a cost function of the solution, J, is set as

$$J = \frac{1}{2}\left(\Delta\theta_1^2 + \Delta\theta_2^2 + \Delta\theta_3^2 + \Delta\theta_4^2\right) \tag{23}$$

If the variable corresponding to the arbitrary parameter is assumed to be $\Delta\theta_4$, the following computation is used in the Newton method, where the nth value of the parameter is expressed as $\Delta\theta_{4n}$ and the initial value of the parameter, $\Delta\theta_{40}$, is set at 0:

$$\Delta\theta_{4(n+1)} = \Delta\theta_{4n} - \left.\frac{J'}{J''}\right|_{\Delta\theta_4 = \Delta\theta_{4n}} \tag{24}$$

Here, $'$ reveals the derivative with respect to parameter $\Delta\theta_4$, that is,

$$J' = \frac{dJ}{d\Delta\theta_4}, \quad J'' = \frac{d^2J}{d\Delta\theta_4^2}$$

In the computation of the Newton method, J' and J'' are necessary to be calculated. They are obtained from the Gröbner bases, g_3, g_2, and g_1, as follows: Firstly, g_3, g_2, and g_1 are differentiated by $\Delta\theta_4$ as

$$\begin{bmatrix}0 & 0 & g_{33}\\ 0 & g_{22} & g_{23}\\ g_{11} & 0 & g_{13}\end{bmatrix}\begin{bmatrix}\Delta\theta_1'\\ \Delta\theta_2'\\ \Delta\theta_3'\end{bmatrix} = -\begin{bmatrix}g_{34}\\ g_{24}\\ g_{14}\end{bmatrix} \tag{25}$$

where g_{ij} expresses the derivative of g_i by $\Delta\theta_j$. By solving the above equation, $\Delta\theta_1'$, $\Delta\theta_2'$, and $\Delta\theta_3'$ are obtained. From these values, J' is obtained as

$$J' = \Delta\theta_1\Delta\theta_1' + \Delta\theta_2\Delta\theta_2' + \Delta\theta_3\Delta\theta_3' + \Delta\theta_4 \tag{26}$$

Similarly, the following equation is obtained by differentiating g_1, g_2, and g_3 with respect to $\Delta\theta_4$ twice:

$$\begin{aligned}&\begin{bmatrix}0 & 0 & g_{33}\\ 0 & g_{22} & g_{23}\\ g_{11} & 0 & g_{13}\end{bmatrix}\begin{bmatrix}\Delta\theta_1''\\ \Delta\theta_2''\\ \Delta\theta_3''\end{bmatrix}\\ &= -\begin{bmatrix}g_{344} + 2g_{334}\Delta\theta_3' + g_{333}{\Delta\theta_3'}^2\\ g_{244} + 2g_{224}\Delta\theta_2' + 2g_{234}\Delta\theta_3' + g_{233}{\Delta\theta_3'}^2\\ g_{144} + 2g_{114}\Delta\theta_1' + 2g_{134}\Delta\theta_3' + g_{133}{\Delta\theta_3'}^2\end{bmatrix}\end{aligned} \tag{27}$$

where g_{ijk} means the second partial derivative of g_i with respect to $\Delta\theta_j$ and $\Delta\theta_k$. Note that $g_{222} = g_{223} = g_{311} =$

$g_{313} = 0$ by this definition By solving the above equation, $\Delta\theta_1''$, $\Delta\theta_2''$, and $\Delta\theta_3''$ are obtained. From these values, J'' is obtained as

$$J'' = \Delta\theta_1'^{\,2} + \Delta\theta_1\Delta\theta_1'' + \Delta\theta_2'^{\,2} + \Delta\theta_2\Delta\theta_2'' + \Delta\theta_3'^{\,2} + \Delta\theta_3\Delta\theta_3'' + 1 \quad (28)$$

By obtaining the values of J' and J'', the Newton method can be executed.

4 Numerical Examples

4.1 Passability of singular states

The gimbal angles that express singular states of CMGs are obtained in the following manner[2]: θ_2 and θ_4 are set arbitrarily, and θ_1 and θ_3 are calculated from

$$\tan\theta_1 = -\frac{\cos\beta(2\cos\beta + \tan\theta_2 - \tan\theta_4)}{\tan\theta_2 + \tan\theta_4} \quad (29)$$

$$\tan\theta_3 = -\frac{\cos\beta(2\cos\beta - \tan\theta_2 + \tan\theta_4)}{\tan\theta_2 + \tan\theta_4} \quad (30)$$

In the case of the pyramid-type arrangement of four CMGs with $\beta = 54.74°$, singular states are shown in Fig. 2 with respect to angular momentum $\boldsymbol{h}$. Here, the value of the angular momentum is divided by h_w. In this figure, the differences in colors express types of the singular states (4H, 2H, and 0H). Among these singular states, impassable singular states by definition 1 are shown in Fig. 3. As shown in this figure, there are impassable singular states within the envelope of the angular momentum. Furthermore, impassable singular states by definition 2 in the direction of the increase of the angular momentum $\boldsymbol{h}$ are shown in 4. From Figs. 3 and 4, the distribution of the impassable singular states is almost the same between two cases. Impassable singular states by definition 2 in the direction of $+z$ are shown in Fig. 5. In this case, the distribution of the impassable singular states is biased toward the region where z-component of the angular momentum has + sign. This result shows that there are many impassable singular states in the direction of the increase of the angular momentum.

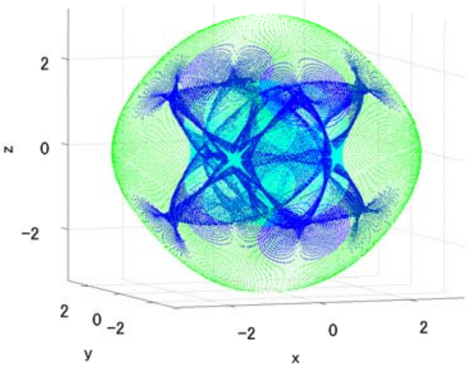


Fig 2: Distribution of singular states

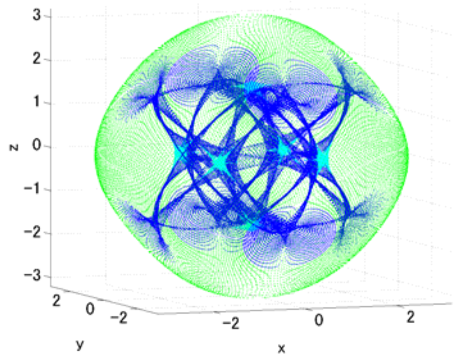


Fig 3: Impassable singular states by definition 1

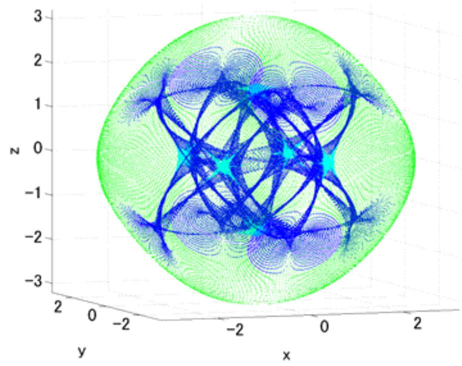


Fig 4: Impassable singular states by definition 2 in the direction of increase of $\boldsymbol{h}$

4.2 Motion of angular momentum and gimbal angles

Here, gimbal angles are calculated from the total angular momentum of CMGs where the angular momentum is increased gradually with $\Delta\boldsymbol{h}$ in the fixed direction from 0 to the value at the envelope. At the non-singular states, $\Delta\boldsymbol{\theta}$ is obtained from the first order of $\Delta\boldsymbol{\theta}$ of Eq. (7) by using the pseudo-inverse matrix of $\boldsymbol{A}$. At the singular states, $\Delta\boldsymbol{\theta}$ is obtained with the consideration of the second-order term of Eq. (7). $\Delta\boldsymbol{\theta}$ is calculated in the following two cases:

$$\text{Case 1} \quad : \quad \Delta\boldsymbol{h} = \begin{bmatrix} h_{xym}/(\sqrt{2}N) & h_{xym}/(\sqrt{2}N) & 0 \end{bmatrix}^T$$

$$\text{Case 2} \quad : \quad \Delta\boldsymbol{h} = \begin{bmatrix} h_{xm}/N & 0 & 0 \end{bmatrix}^T$$

where h_{xym} and h_{xm} are the maximum angular momentum at the envelope in the direction of $x = y > 0$ and $x > 0$, respectively. In the case of the pyramid-type arrangement of four CMGs with $\beta = 54.74°$, the relation between the angular momentum of CMGs and the gimbal angles in case 1 is shown in Fig. 6 where N is set at 500. In this figure, the total number of $\Delta\boldsymbol{h}$ divided by N is taken as abscissa. Figure (a) shows the gimbal angles of CMGs and figure (b) shows the calculated total angular momentum of CMGs from the gimbal angles in figure (a). The passable and impassable singular states by definition 2 are shown by $\circ$ and $\square$, respectively, where

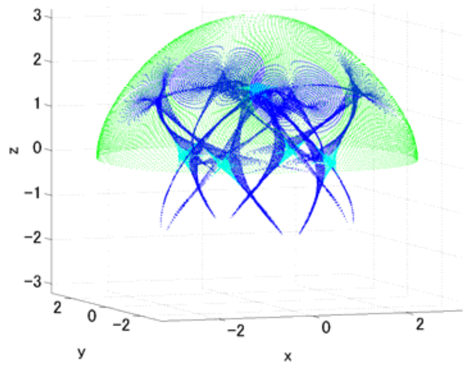


Fig 5: Impassable singular states by definition 2 in the direction of $+z$

the state is judged as singular if the condition number of $\boldsymbol{A}$ is larger than 20. In case 1, the states remain singular for a while with the increase of the angular momentum, where all the singular states are passable. The relation between the angular momentum of CMGs and the gimbal angles in case 2 is shown in Fig. 7. In this case, only one state becomes singular, where this singular state is impassable. The gimbal angles change discontinuously before and after the singular state. Because the values of $\Delta\boldsymbol{\theta}$ are not small, there occur some errors in $\Delta\boldsymbol{h}$ in Eq. (7). Therefore, small discontinuity in the angular momentum is observed before and after the singular state in this case.

4.3 Attitude control example

Next, attitude control examples are shown by using the solution of $\Delta\boldsymbol{\theta}$ from the second-order equation (7). By focusing on the case where the angular momentum of CMGs passes the impassable singular state as shown in case 2, two cases are compared here; one by the general treatment of singular states and one by using the solution of Eq. (7). The equations of motion of the spacecraft are expressed as

$$\boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{h}_T = -\boldsymbol{A}\dot{\boldsymbol{\theta}} \tag{31}$$

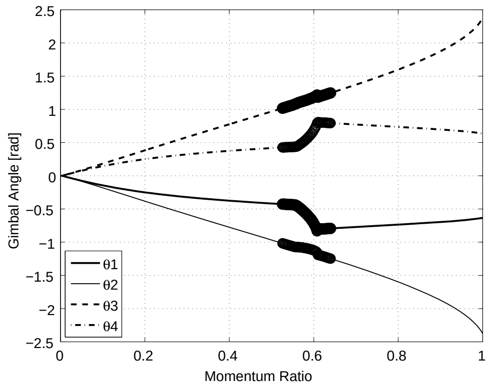
where $\boldsymbol{J}$ is the inertia tensor of the spacecraft, $\boldsymbol{\omega}$ is the angular velocity of the spacecraft, and $\boldsymbol{h}_T$ is the total angular momentum of the spacecraft. These tensor and vectors are expressed in the body-fixed coordinates. The control torque of the spacecraft, $\boldsymbol{\tau}_c$, has the relation with $\dot{\boldsymbol{\theta}}$ as

$$\boldsymbol{\tau}_c = -\boldsymbol{A}\dot{\boldsymbol{\theta}} \tag{32}$$

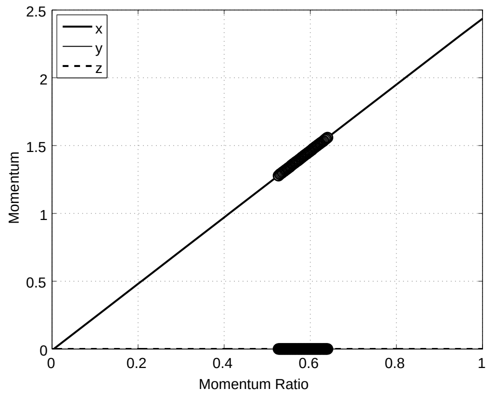
The gimbal rates $\dot{\boldsymbol{\theta}}$ are determined to satisfy the above equation. If the CMGs are not in the singular state, $\dot{\boldsymbol{\theta}}$ is calculated as follows:

$$\dot{\boldsymbol{\theta}} = -\boldsymbol{A}^{\dagger}\boldsymbol{\tau}_c \tag{33}$$

where $\boldsymbol{A}^{\dagger}$ is a pseudo-inverse matrix of $\boldsymbol{A}$. In the case where $\boldsymbol{A}$ becomes singular, the following two control laws are considered:



(a) Gimbal angles



(b) Angular momentum of CMGs

Fig 6: Gimbal angles and angular momentum in case 1

4.3.1 Control 1

The singular value decomposition of $\boldsymbol{A}$ is expressed as

$$\boldsymbol{A} = \boldsymbol{U}\begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{bmatrix}\boldsymbol{V}^T \tag{34}$$

where σ_1, σ_2, and σ_3 are the singular values of $\boldsymbol{A}$ with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. At singular states, $\dot{\boldsymbol{\theta}}$ is given as

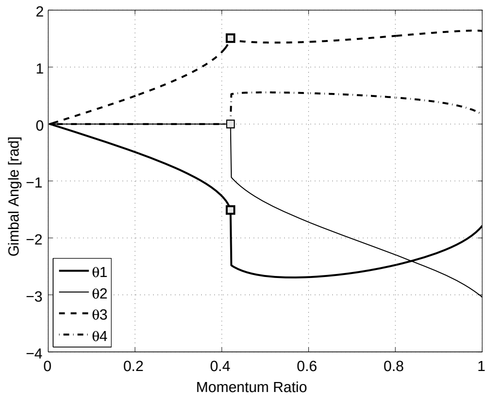
$$\dot{\boldsymbol{\theta}} = -\boldsymbol{V}\begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 1/(\sigma_3+\lambda) \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{U}^T\boldsymbol{\tau}_c \tag{35}$$

where λ is a proper positive number. This control law is called the singular direction avoidance steering law[3].

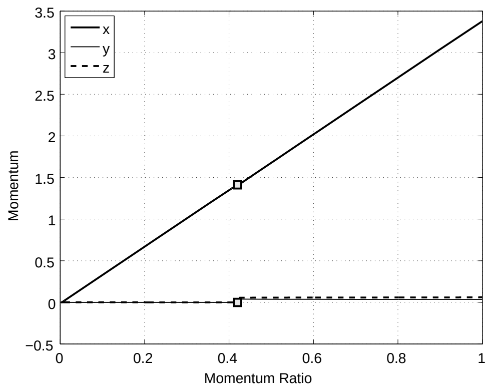
4.3.2 Control 2

By choosing an appropriate small time Δt, $\Delta\boldsymbol{h}$ is set as $\Delta\boldsymbol{h} = -\boldsymbol{\tau}_c\Delta t$. $\Delta\boldsymbol{\theta}$ is calculated to satisfy Eq. (7). Then, $\dot{\boldsymbol{\theta}}$ is obtained by

$$\dot{\boldsymbol{\theta}} = \frac{\Delta\boldsymbol{\theta}}{\Delta t} \tag{36}$$



(a) Gimbal angles



(b) Angular momentum of CMGs

Fig 7: Gimbal angles and angular momentum in case 2

In these two control laws, the magnitude of $\dot{\boldsymbol{\theta}}$ is limited to a certain value. That is, when $\left|\dot{\boldsymbol{\theta}}\right| > \dot{\theta}_{max}$, $\dot{\boldsymbol{\theta}}$ is replaced by

$$\frac{\dot{\theta}_{max}}{\left|\dot{\boldsymbol{\theta}}\right|}\dot{\boldsymbol{\theta}}$$

4.3.3 Simulation results

Parameters used in the simulations are

$$\boldsymbol{J} = \begin{bmatrix} 500 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 500 \end{bmatrix} \ [\mathrm{kgm}^2],$$

$$h_w = 10 \ [\mathrm{Nms}], \quad \boldsymbol{h}_T = \boldsymbol{0},$$

$$\Delta t = 0.1 \ [\mathrm{s}], \quad \dot{\theta}_{max} = 5 \ [\mathrm{rad/s}]$$

The attitude control is executed so that the angular momentum of the spacecraft body is increased in the $-x$ direction (the angular momentum of CMGs is increased in the $+x$ direction) before and after the singular state shown in Fig. 7. The angular momentum of the spacecraft body is increased from 0 to 5 [s], whereas it is kept constant after 5 [s]. If the condition number of $\boldsymbol{A}$ is larger 20, the state is judged as singular, and then, control 1 or control 2 is adopted.

The attitude control result by control 1 with $\lambda = 5$ is shown in Fig. 8. These figures show the gimbal angles, gimbal rates, and attitude control error between the target value of the attitude control corresponding to the angular momentum of the spacecraft body and the spacecraft attitude, from the top. Attitude control error is expressed in terms of the vector part of Euler parameters between the target value and the real value. In this case, CMGs become singular at $t = 0.9$[s], and the attitude control torque is generated a little from $t = 0.9$[s] to $t = 4$[s]. After $t = 4$[s], the gimbals move rapidly due to the accumulation of the attitude control error, and CMGs escape from the singular state at $t = 4.4$[s]. After that, the attitude control error is decreased by the attitude control. On the other hand, the simulation result by control 2 is shown in Fig. 9. In this case, CMGs also become singular at $t = 0.9$[s]. However, they escape from the singular state at $t = 1.1$[s] by virtue of the attitude control with the solution of Eq. (7). The attitude control error is also suppressed small during the singular states.

5 Conclusions

In this paper, singular states of a pyramid-type arrangement of four single-gimbal CMGs are considered. The passability of the singular states is newly defined by using the second-order terms of the small gimbal angles, and the passability condition based on the definition is introduced. The computation method for the small changes of the gimbal angles with the consideration of the second-order terms is proposed, and it is applied to the attitude control at the singular state of CMGs. The validity of the attitude control is verified by the numerical simulations.

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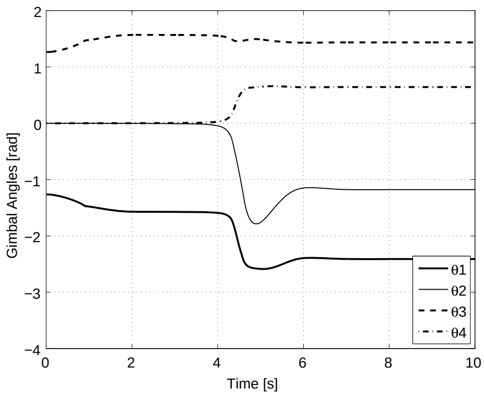
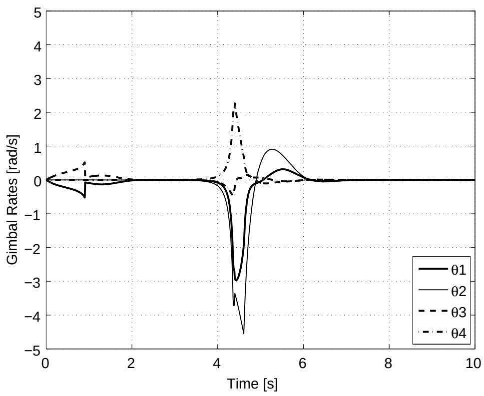
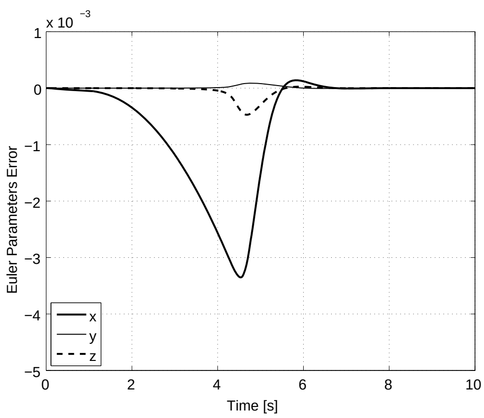


(a) Gimbal angles

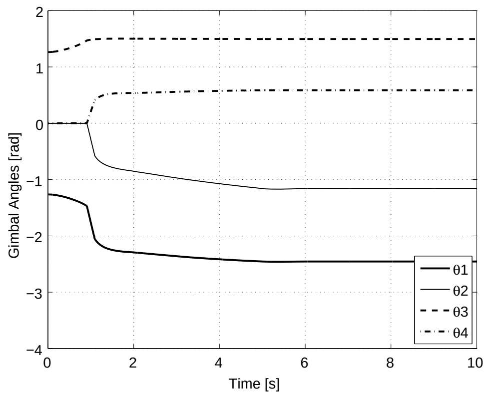


(b) Gimbal rates

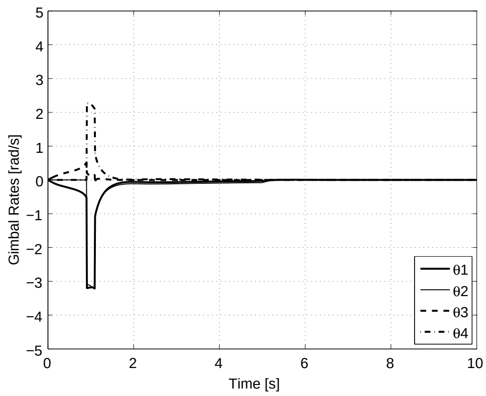


(c) Attitude control error

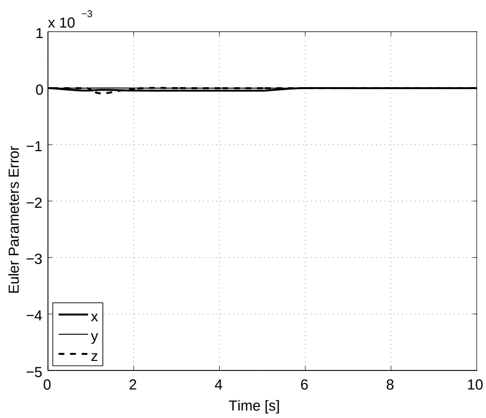
Fig 8: Gimbal angles and attitude control error in control 1



(a) Gimbal angles



(b) Gimbal rates



(c) Attitude control error

Fig 9: Gimbal angles and attitude control error in control 2