Composition of Physical Rotations and Its Rotation Representation

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Abstract: A physical value express characteristic of a physical object and does not depend on the coordinates with which it is described. But a coordinate system is needed for showing it in numbers. So coordinate transformations are used often in computation of physical values. Meanwhile a physical rotation of a rigid body is represented using the transformation of the coordinates fixed on it. These transformations are frequently confused.

This paper shows their difference and the composition of physical rotations using direction cosine matrices.

物理量としての回転の合成とその回転表現

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摘要 : 物理量は測定対象に固有な物理系の性質を記述する量で座標系に依存しない。しかし、物理量であるベクトルや回転を具体的な数値として表すためには座標系を用いる必要がある。そのため、物理量の計算にしばしば座標変換が使われる。一方物理的な回転は、剛体に固定された座標系の変換としてモデル化される。これら２つの変換は混同されることがある。

ここでは、これらの変換の持つ性質の違いを示し、方向余弦行列を用いて物理的な回転の合成の具体的な計算式を示す。

1. Introduction

Physical motion consists of translational and rotational motion.

Translational motion is described by force, acceleration, velocity, and position. They are vectors and are calculated under the rules of vectors. Rotational motion is described by torque, angular acceleration, angular velocity, and rotation. Torque, angular acceleration, and angular velocity are vectors.

A rotation is a physical value concerning rotational motion, which corresponds to position in translational motion but it is not a vector. It is an element of a kind of group and is calculated under other rules.

Generally, a physical value express characteristic of a physical object and does not depend on the coordinate frame with which it is described. But a coordinate system is needed for showing it in numbers[1,2). Therefore coordinate transformation is used often in computation of vectors, while the physical rotation of a rigid body is represented as the coordinate transformation of a coordinate frame fixed on it.

These transformations are frequently confused[3,4)] but they have different characteristics.

A rotation can be represented by a tensor[5,6,7,8)], but in this paper is shown by a direction cosine matrix[1,3)].

This paper shows the formula of the composition of physical rotations using direction cosine matrices and the properties of the group that is composed by the matrices which represent physical rotations[9,10].

2. Mathematical Coordinate Transformation

Let F be a set of Cartesian coordinates and $E^i = {}^t\left(e^i_1,\ \ e^i_2,\ \ e^i_3\right) \in F$ be its element.

A direction cosine matrix C satisfies the following equations:

$$C^{-1} = {}^tC, \quad \det[C] = 1 \tag{1}$$

A direction cosine matrix C_α transforms a frame $E^1 \in F$ to another $E^2 \in F$, so we can express the relation as follows;

$$E^2 = C_\alpha E^1 \tag{2}$$

If C_β transforms E^2 to E^3, the multiplication of matrices $C_\gamma = C_\beta C_\alpha$ does E^1 to E^3 and C_γ represents also a coordinate transformation.

The set of these matrices as a whole is a group concerning the above calculation of matrices. It is called the rotation group in mathematics.

We denote S_C as the group and call its element as a mathematical coordinate transformation.

3. Composition of physical rotations

The starting point for describing a physical rotation is Euler's Theorem: "The general displacement of a rigid body with one fixed point is a rotation about an axis through that point"[11]. The fixed rotation axis is known as the Euler axis with its unit-vector designated by $\hat{\varepsilon}$. The rotation angle about the Euler axis is represented by the Euler angle θ.

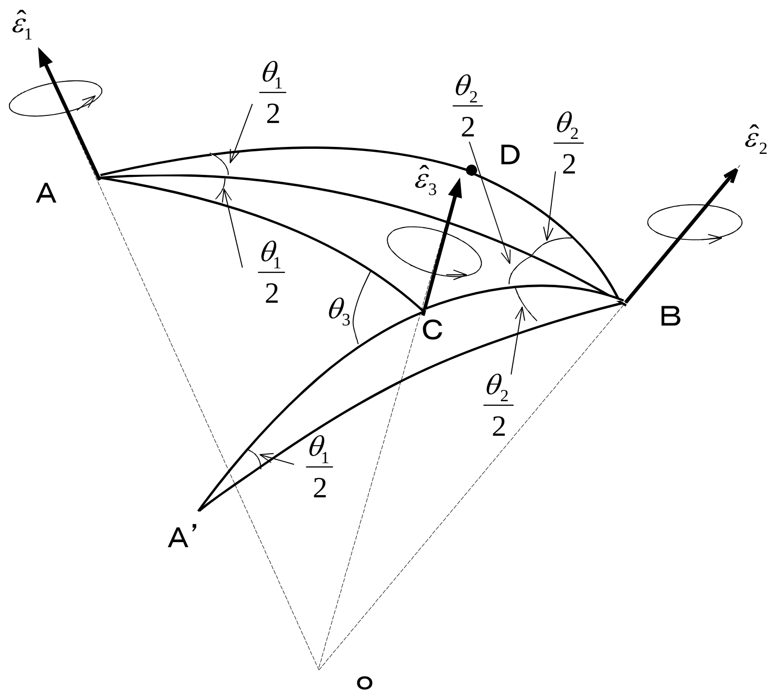

Fig 1. Composition of physical rotations

Let $R(\hat{\varepsilon},\theta)$ be a physical rotation. If rotation $R(\hat{\varepsilon}_2,\theta_2)$ follows rotation $R(\hat{\varepsilon}_1,\theta_1)$, the element $\hat{\varepsilon}_3$ and θ_3 of their composition $R(\hat{\varepsilon}_3,\theta_3)$ can be obtained by geometrical method illustrated as shown in Fig.1 where Points A,B, and C are on a unit sphere[11].

We denote S as the set of physical rotations $R(\hat{\varepsilon},\theta)$ as a whole;

$$S = \{R_q = R(\hat{\varepsilon}_q,\theta_q) \mid q \in P^3\} \tag{3}$$

The composition of two elements R_1 and R_2 of S is an element of S and satisfies the rule of groups, so S is a group.

4. Direction Cosine Matrix of Physical Rotation

Denoting $E^1 = {}^t(e_1^1,\ e_2^1,\ e_3^1)$ as the frame fixed on a rigid body before a physical rotation $R(\hat{\varepsilon},\theta)$, and $E^2 = {}^t(e_1^2,\ e_2^2,\ e_3^2)$ as the frame after it, a direction cosine matrix A is defined as follows:

$$E^2 = AE^1 \tag{4}$$

It is well known that the direction cosine matrix A concerning rotation $R(\hat{\varepsilon},\theta)$ is as follows[1,2,3];

$$A = \begin{pmatrix} \cos\theta + (1-\cos\theta)\varepsilon_1^2 & (1-\cos\theta)\varepsilon_1\varepsilon_2 + (\sin\theta)\varepsilon_3 & (1-\cos\theta)\varepsilon_1\varepsilon_3 - (\sin\theta)\varepsilon_2 \\ (1-\cos\theta)\varepsilon_1\varepsilon_2 - (\sin\theta)\varepsilon_3 & \cos\theta + (1-\cos\theta)\varepsilon_2^2 & (1-\cos\theta)\varepsilon_2\varepsilon_3 + (\sin\theta)\varepsilon_1 \\ (1-\cos\theta)\varepsilon_1\varepsilon_3 + (\sin\theta)\varepsilon_2 & (1-\cos\theta)\varepsilon_2\varepsilon_3 - (\sin\theta)\varepsilon_1 & \cos\theta + (1-\cos\theta)\varepsilon_3^2 \end{pmatrix} \tag{5}$$

where $\varepsilon_1 = (\hat{\varepsilon}\cdot e_1^1),\quad \varepsilon_2 = (\hat{\varepsilon}\cdot e_2^1),\quad \varepsilon_3 = (\hat{\varepsilon}\cdot e_3^1)$ (6)

This matrix depends on the coordinates $E^1 = {}^t(e_1^1,\ e_2^1,\ e_3^1)$, so we denote $C(R,E^1)$ as the matrix concerning rotation $R(\hat{\varepsilon},\theta)$, which is called as a physical rotation matrix in this paper.

Let a physical rotation matrix be $C(R,E^0)$ where E^0 is a reference frame.

We can get a physical rotation matrix $C(R,E^1)$ as follows.

Let a direction cosine matrix C_α be the mathematical coordinate transformation which moves the frame E^0 to the frame E^1. The next equation satisfied.

$$E^1 = C_\alpha E^0 \tag{7}$$

If we suppose the frames E^0 and E^1 are fixed on the same rigid body, the result of a physical rotation $R(\hat{\varepsilon},\theta)$ is represented respectively as follows;

$$E'^0 = C(R,E^0)E^0 \tag{8}$$

$$E'^1 = C(R,E^1)E^1 \tag{9}$$

The relative relation of these frames is not changed by the rotation, so the following equations are satisfied.

$$E'^1 = C_\alpha E'^0 \tag{10}$$

$$E'^1 = C_\alpha C(R, E^0)E^0 = C_\alpha C(R, E^0)(C_\alpha)^{-1}E^1 \tag{11}$$

$$\therefore \quad C(R, E^1) = C_\alpha C(R, E^0)(C_\alpha)^{-1} \tag{12}$$

Let S_C be the set of physical rotation matrices as a whole;

$$S_C = \{C(R_q, E^0) \mid R_q \in S\} \tag{13}$$

Let $R_1 \circ R_2$ be the composition of R_1 and R_2, if a physical rotation R_2 follows a rotation R_1.

If E^0 is transformed to E^1 by a physical rotation R_1, the next equation is satisfied.

$$E^1 = C(R_1, E^0)E^0 \tag{14}$$

And if E^1 is done to E^2 by a physical rotation R_2, the next equation is satisfied.

$$E^2 = C(R_2, E^1)E^1 = C(R_2, E^1)C(R_1, E^0)E^0 = C(R_1 \circ R_2, E^0)E^0 \tag{15}$$

Using Eq.(12), the next equation is obtained.

$$C(R_1 \circ R_2, E^0) = C(R_2, E^1)C(R_1, E^0) = C_\alpha C(R_2, E^0)(C_\alpha)^{-1}C(R_1, E^0) \tag{16}$$

In this case the next equation is satisfied using Eqs.(7) and (14).

$$C_\alpha = C(R_1, E^0) \tag{17}$$

So we get the next equation.

$$C(R_1 \circ R_2, E^0) = C(R_1, E^0)C(R_2, E^0) \tag{18}$$

These relations are shown in Fig.2.

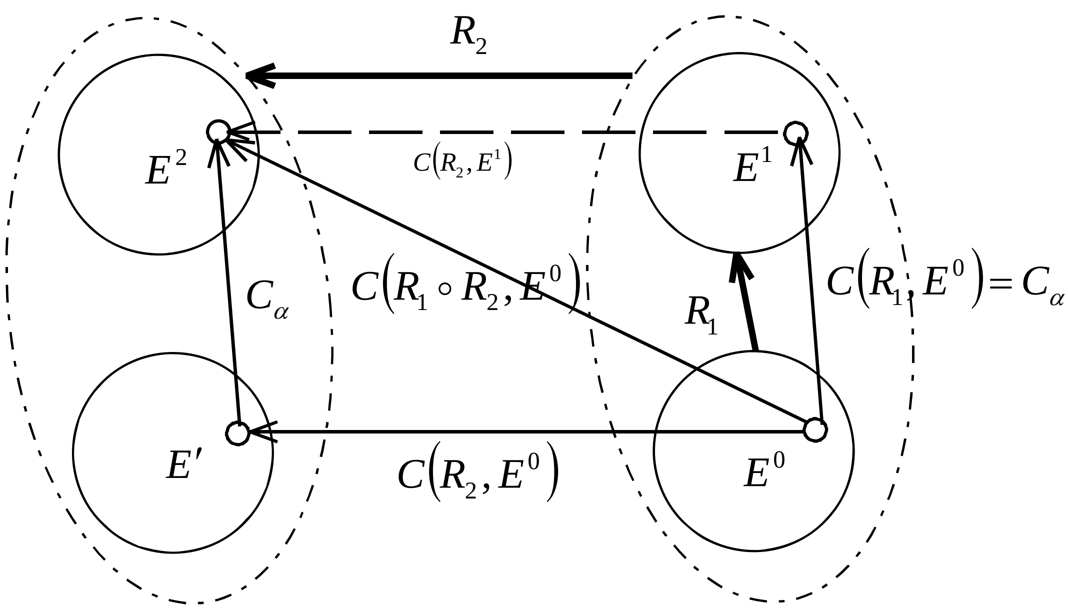

Fig2. Composition of physical rotation matrices

The set S_C of physical rotation matrices on the reference frame E^0 is a group.
A function $h: S \to S_C$ can be defined as follows;

$$C\left(R_q, E^0\right) = h\left(R_q\right) \tag{19}$$

Each $C\left(R_q, E^0\right) \in S_C$ corresponds to the physical rotation $R_q \in S$ and a product of matrices corresponds to a composition of physical rotations. So the group S_C is isomorphic to the group S of physical rotations.
As the next equation is not always satisfied;

$$C\left(R_1, E^0\right)C\left(R_2, E^0\right) = C\left(R_2, E^0\right)C\left(R_1, E^0\right) \tag{20}$$

the group S_C is unexhangeable.
Generally any rotation representation can correspond to its rotation matrix, so the methods of their compositions can be derived using Eq.(18).

5. Conclusion

A mathematical coordinate transformation can be represented as a direction cosine matrix C_α.
If a transformation C_β follows to C_α, the composition $C_\alpha \circ C_\beta$ of them is represented by the multiplication of matrices as follows.

$$C_\alpha \circ C_\beta = C_\beta C_\alpha \tag{21}$$

The set of matrices which correspond to mathematical coordinate transformations as a whole is a group using Eq.(21).
Rotational motion is described by torque, angular acceleration, angular velocity, and rotation. Torque, angular acceleration, and angular velocity are vectors but rotation is not a vector. It is an element of a kind of group and is calculated under the rule of a group.
The rotational motion of a rigid body can be represented the motion of the coordinates fixed on it. So a physical rotation corresponds to a coordinate transformation, which is represented also using a direction cosine matrix.
The direction cosine matrix $C\left(R, E^0\right)$ which represents a physical rotation depend on the reference frame.
If a physical rotation R_2 follows R_1, the matrix which corresponds to the composition $R_1 \circ R_2$ is represented as follows.

$$C\left(R_1 \circ R_2, E^0\right) = C\left(R_1, E^0\right)C\left(R_2, E^0\right) \tag{22}$$

The set of matrices which correspond to physical rotations as a whole is a group using Eq.(22). This group is isomorphic to that of physical rotations.
The group of mathematical coordinate transformations correspond to that of physical rotations. But their calculation method is different.

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