Earth Escape from a Sun-Earth Halo Orbit Using the Unstable Manifold and Lunar Gravity Assists

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This paper investigates the Earth escape for the spacecraft in a Sun-Earth halo orbit. The escape trajectory consists of first ejecting to the unstable manifold associated to the halo orbit, then coasting along the manifold and last performing lunar gravity assisted escape. The first intersection of the manifold tube and Moon’s orbit results in four intersection points. These four manifold-guided encounters have different relative velocities (υₗ) to the Moon; therefore, the corresponding lunar swingbys can result in different levels of characteristic energy (C₃) with respect to the Earth. To further exploit these manifold-guided lunar encounters, subsequent swingbys considered. A graphical method is introduced to reveal the theoretical upper limits of the expectable C₃ using double and multiple swingbys. The Moon-to-Moon solutions indicate that a second lunar swingby can efficiently increase C₃. Comparing with the direct low-energy escape along manifold, applying a portion of lunar swingbys before escape is shown more advantageous for deep-space mission design.

Key Words: circular restricted three-body problem; invariant manifolds; lunar gravity assists; Earth escape; graphical method

1. Introduction

The periodic halo orbit about the equilibrium point of a three-body system (also known as the libration point or Lagrangian point) has features such as relatively constant distances and orientation with respect to the primary and secondary bodies, which are advantageous for scientific observation and spacecraft operation. There have been several successful missions around the Sun-Earth libration points L₁ and L₂ and many being planned. Given a small ΔV, the spacecraft in a halo orbit will depart away along the unstable manifold associated with the orbit. Inversely, a spacecraft can asymptotically converge into the halo orbit along the associated stable manifold. Utilizing the manifold dynamics, interesting missions can be derived from the libration point region. In particular, low-energy escape is of great interests, and has been intensively studied [1–3]. The low-energy escape requires lower cost than that in the two-body model. The fuel saving in escaping can contribute to an increase of the payload mass. However, the energy of the spacecraft after low-energy escape only permits accessing near-earth-orbit asteroids within a narrow zone around Earth’s orbit. Ref. [2] and [3] on the transfer from the Earth to Mars have to apply a ΔV of around 2 km/s after the low-energy escape from the Sun-Earth halo orbits. To gain more possibility and flexibility of interplanetary missions, escape with high energy is desirable.

Lunar swingbys have proven effective in increasing escape energy in some mission design and analysis [4][5][6][7][8]. In particular, if a longer flight time is permitted, the solar perturbation can be utilized to vary the Vₖ with respect to the Moon, greatly enhancing the flexibility of mission design. This strategy can be applied to improving the lunar encounter condition for gravity assisted escape, or reducing the required insertion Δv for a lunar mission. The mission design methods of HITEN, PLANET-B, LUNAR-A and ARTEMIS have demonstrated this technique.[5, 9, 10]

This study aims to give an analysis of the Earth escape for the spacecraft initially in a Sun-Earth L₁/L₂ halo orbit. It is also motivated by the JAXA mission DESTINY (Demonstration and Experiment of Space Technology for Nearplanetary voYage) which will be launched in 2019 and go to a Sun-Earth L2 halo orbit using solar electric propulsion (SEP)[11]. The possibility of extending the mission to visit a heliocentric body is currently under discussion. In order to gain high escape energy for flexible mission design as well as broad achievable domain, this paper applies manifolds for leaving the halo orbit, and then lunar swingbys for gaining energy before escape.

The following discussion is organized into three parts. Sec. 2 presents the manifold-guided lunar encounters. The Earth escape trajectories derived from the corresponding lunar swingbys are compared with the direct escape along the anti-Sun-ward manifold in terms of the escape energy C₃ with respect to the Earth. In order to further increase C₃, Sec. 3 considers using multiple lunar swingby aided by solar perturbation to further increase C₃. First, a graphical method is introduced to analyze the lun-i-solar gravity-assisted Earth escape, revealing theoretical upper limits of the C₃ achieved by double, triple and multiple lunar swingbys. As considerable C₃ is shown expectable by a second swingby in the graphical analysis,
Sun-perturbed Moon-to-Moon transfers are numerically solved for four types of manifold-guided encounter. Results indicate the practically achievable $C_3$, required flight time and escape directions. Conclusions and discussions are given in the last part.


2.1. Invariant manifolds

Trajectories are considered in the circular restricted three-body problem (CR3BP) of the Sun-Earth system. The equations of motion in the Sun-Earth synodic frame are given in Appendix A. There are five equilibrium points, among which the three collinear points $L_1$, $L_2$ and $L_3$ are unstable. The linearized equations at $L_1$, $L_2$ and $L_3$ reveal that there exist unstable periodic orbits about these points. The monodromy matrix is the state transition matrix (STM) after one period of a periodic orbit. By examining the eigenvalues and eigenvectors of the monodromy matrix, one can know the characteristics of the orbit. For halo orbits, there is a pair of eigenvalues with $\lambda_1, \lambda_2 \approx 1$. The dominant eigenvalue $\lambda_2$ is around 1500 for the Sun-Earth $L_1/L_2$ halo orbits. A small displacement along the eigenvector of the dominant eigenvalue (divergent eigenvector or unstable subspace for short) will propagate into exponentially increasing divergence from the halo orbit, since it grows $\lambda_2$ times in a period. Such a departure trajectory is along the invariant unstable manifold associated with halo orbit. The derived manifold trajectory can go Sun-ward or anti-Sun-ward. Previous studies on low-energy escape utilize the anti-Sun-ward manifolds [2][3]. In order to apply lunar gravity assists, this paper focuses on the Sun-ward manifolds. The propagation of the perturbed states yields a tube structure of the unstable manifold. Fig. 1 shows the Sun-ward unstable manifold trajectories for the Sun-Earth $L_2$ halo orbit with a $z$-amplitude ($A_z$) of $4\times10^3$ km.

2.2. Manifold-guided lunar encounters

This section attempts to find the manifold trajectories that encounter Moon's orbit. In this paper, the spacecraft is assumed initially in a Sun-Earth $L_2$ halo orbit. The inclination and eccentricity of Moon's orbit are small and assumed zero, as general intersections of unstable manifolds with Moon's orbit are considered regardless of the ephemeris. In addition, the Moon's gravity is considered only at lunar encounter where it impulsively deflects the $v_\infty$ with respect to the Moon. Although it is a simplified model, reliable insights can still be acquired for mission design.

In Fig. 1, grey circles mark the intersections of the manifold trajectories and the elliptic plane, outlining the intersection lines of the manifold tube and the elliptic plane. It can be seen that there are four intersections of the manifold tube and Moon's orbit. Moreover, for a wide $A_z$ range investigated (from $1\times10^3$ km to $5\times10^3$ km), the four types of intersection can be found. Note that the four intersections are found at the first crossing through Moon's orbit. There will be more intersections that take place at the second and subsequent crossings through Moon's orbit, which would take longer flight time and are not discussed in this paper. The four manifold trajectories intersecting the Moon's orbit are obtained via interpolation. They are colored in the figures. The four types of intersection are numbered in order of the lunar phases.

Fig. 1. The crossing of the manifold tube and Moon's orbit exhibits four types of intersection (Sun-Earth synodic frame).

Supposing the Moon will be at the intersection points when the manifold trajectories arrive at Moon's orbit, the relative velocity ($v_\infty$) to the Moon determines the lunar gravity assist capacity. Because the Jacobi integral does not change too much due to small departure $\Delta V$, distances from Moon's orbit to the Earth are identical and the Sun is relatively far, the manifold trajectories have comparable velocities with respect to the Earth at lunar encounters. However, due to the different approaching directions with respect to Moon's orbit, the $v_\infty$ differ greatly. The $v_\infty$ of the four types of lunar encounter are given in Fig. 2 as a function of $A_z$. The manifold trajectories of type 1 and type 2 are shown mostly tangential to Moon's orbit while the manifold trajectories of type 3 and type 4 are mostly perpendicular. As a result, the encounters of type 3 and type 4 result in larger $v_\infty$ (around 1.35 km/s) than the encounters of type 1 and type 2. The $v_\infty$ of type 3 and the $v_\infty$ of type 4 are nearly equal (the two lines in the figure overlap) and do not vary too much with the size of halo orbit. The $v_\infty$ of type 1 and the $v_\infty$ of type 2 are comparable. They are up to 0.6 km/s at $A_z = 5\times10^3$ km, and decrease relatively greatly as $A_z$ decreases.
2.3. Earth escape facilitated by the lunar swingbys

This section investigates the capacity of the four lunar encounters for gravity-assisted Earth escape. The $C_3$ is usually used to describe the escape energy. It is twice the specific energy $e$; that is,

$$C_3 = 2e = v^2 - 2\mu_E/r$$

where $\mu_E$ is the gravitational parameter of the Earth, and $v$ and $r$ are the velocity and position with respect to the Earth. For brevity, unless noted, the $C_3$ presented hereafter refers to the $C_3$ with respect to the Earth, and the $v_e$ refers to the $v_e$ with respect to the Moon. As $C_3$ can be influenced by the solar perturbation, $C_{3\text{LSB}}$ is used to represent the osculating $C_3$ at the lunar swingby. The radius of Moon’s orbit $r_m$ is assumed constant (= 384,400 km). In order to get the maximum post-swingby $C_{3\text{LSB}}$, the post-swingby velocity $v'$ should be maximized (superscripts “$+$” and “$-$” indicate the states prior to and after a swingby, respectively). A lunar swingby will rotate the incoming $v_z^+$ by an angle $\delta$ to an outgoing $v_z^-$. The bending angle $\delta$ is associated to the perilune altitude that can be targeted by a small maneuver at a far distance before swingby. In order to acquire $v_{\text{max}}^+$, the outgoing $v_z^-$ should be aligned with the Moon’s velocity vector as much as possible. However, $\delta$ is limited to the $\delta_{\text{max}}$ expressed by

$$\delta_{\text{max}} = \pi - 2\arccos\left[\mu_M\left(\mu_M + r_{\text{min}}v_{z}^2\right)\right]$$

where $\mu_M$ is the gravitational parameter of the Moon, $r_{\text{min}}$ is the minimum perilune, i.e. the sum of the Moon’s radius 1738 km and the minimum permissible flyby altitude, which is 100 km in this study. If the pump angle $\varphi$ between the $v_z^+$ and the Moon’s velocity $v_M$ is smaller than $\delta_{\text{max}}$, $v_{\text{max}}^+$ is the sum of the $v_M$ and $v_e$ magnitudes. Otherwise, to acquire $v_{\text{max}}^+$, the $v_z^-$ should be turned by $\delta_{\text{max}}$ to approach $v_M$ in the plane determined by $v_z^+$ and $v_M$. The two situations are illustrated in Fig. 3. The expression of $v_{\text{max}}^+$ is

$$v_{\text{max}}^+ = \frac{v_M + v_e}{\sqrt{v_M^2 + v_e^2 + 2v_Mv_e\cos(\varphi - \delta_{\text{max}})}}$$

Then, one can acquire the capacity of the lunar encounter for gravity-assisted Earth escape, which is expressed by,

$$C_{3\text{LSBmax}} = v_{\text{max}}^+ - 2\mu_E/r_M$$

Fig. 3. Use lunar swingbys to bend the $v_e$ to achieve the maximum post-swingby velocities.

The $C_{3\text{LSBmax}}$ for each type of lunar swingby as a function of $A_z$ is shown in Fig. 4. As explained earlier, the velocities of the manifold trajectories with respect to the Earth at lunar encounters are comparable. For comparison, the average of the four $C_{3\text{LSB}}$ (i.e. $C_1$ before lunar swingby) is also displayed in the figure for comparison. Without lunar swingbys, the manifold trajectories are considered non-escape as $C_{3\text{LSB}}$ are below zero. Similar to the profiles of $v_e$, the $C_{3\text{LSBmax}}$ of type 3 and type 4 do not vary too much with $A_z$, around 2.6 km$^2$/s$^2$. The $C_{3\text{LSBmax}}$ of type 1 and type 2 are much smaller, with 0.5 km$^2$/s$^2$ at $A_z = 5\times10^5$ km. Furthermore, the swingbys of type 1 and type 2 cannot lead to Earth escape when $A_z$ is smaller than $3\times10^5$ km (the solar perturbation might change the escape situation slightly; see next subsection).

Fig. 4. $C_{3\text{LSB}}$ and $C_{3\text{LSBmax}}$ of the four types of lunar swingby. (The lines of type 3 and type 4 overlap.)

2.4. Comparison with the direct escape along anti-Sun-ward manifolds

As has been mentioned, by varying the bending angle $\delta$ within $\delta_{\text{max}}$, one can get various post-swingby states as well as different $C_{3\text{LSB}}$. However, the Sun also influences the trajectory, especially at the edge of Earth sphere of influence. A trajectory with low positive $C_{3\text{LSB}}$ could be decelerated and cannot escape for a long time. To examine escape directions, the post-swingby states are propagated until the Earth’s gravity is negligible. For this purpose, a sphere of escape judgment (SoE) centered at the Earth is defined with a radius of 0.02 AU (approximately two times the Earth-L$_2$ distance). At this distance, the Earth’s gravity is less than one tenth of the Sun’s. If a post-swingby trajectory cannot reach the SoE in a

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1 Astronomical Unit, roughly the Sun-Earth distance, i.e. 1 AU = 1.5x10$^8$ km
sufficiently long time (60 days in the study), it is considered non-escape. Furthermore, the escape due to the lunar gravity assists can be compared with the direct escape along the anti-Sun-ward unstable manifold. In the following, the halo orbit with the $A_i$ of $4 \times 10^3$ km is taken as an example for further discussion. The $C_1$ magnitudes at the SoE, $C_{3\text{LSB}}$, and escape directions represented by longitudes and latitudes with respect to the Sun-Earth ecliptic coordinate are printed in Fig. 5. The enclosed solid circle in the figure indicates the direct escape along the anti-Sun-ward manifold. The dots indicate post-swingby trajectories which are acquired by varying the swingby bending angle. The $C_{3\text{LSB}}$ of direct manifold escape ranges from 0.3 to 0.7 km$^2$/s$^2$. The maximum $C_{3\text{LSB}}$ of the post-swingby trajectories is 3.3 km$^2$/s$^2$. The increase from the maximum $C_{3\text{LSB}}$ at $\alpha = 8 \times 10^3$ km is due to the solar perturbation. Higher $C_{3\text{LSB}}$ suggests a larger reachable domain in deep space as well as more accessible targets. The lunar swingbys lead to a wider range of escape directions as well.

3. Multiple Lunar Swingbys for Earth Escape

In the preceding section, the lunar swingbys of type 1 and type 2 are found ineffective for Earth escape. In order to sufficiently utilize these manifold-guided lunar encounters, subsequent lunar swingbys are considered. Under the two-body model, multiple lunar swingbys have invariant $v_\infty$ with respect to the Moon. However, the solar gravitational perturbation (also known as the solar tidal force) can vary $v_\infty$ if the loop between the two lunar swingbys reaches to the region solar perturbation is significant. The solar tidal force in the Earth-centered inertial frame and its effect on orbits around the Earth are schematically depicted in Fig. 6. As illustrated, after the first lunar swingby (S1), a consequent orbit with its apogee far away in the 1st or 3rd quadrants will experience posigrade deceleration near the apogee, and come back to a second lunar swingby (S2) with a larger encounter angle with respect to the Moon or even in a retrograde direction, resulting in an increased $v_\infty$. On the contrary, an orbit with its apogee in the 2nd or 4th quadrants will be accelerated, resulting in a decreased $v_\infty$ at S2 or missing Moon’s orbit. It is observed that, when the osculating apogee $r_a$ at S1 is greater than $8 \times 10^3$ km, the solar perturbation can significantly change the shape of the post-swingby orbit as well as the re-encounter $v_\infty$.

Sec. 3.1 presents a graphical analysis. The graph shows insights into the influence of lunar swingby and Sun-perturbed transfer on the capacity of a subsequent lunar swingby for Earth escape in terms of $C_{3\text{LSB}}$. The graphical analysis will reveal (a) the upper limit of the $C_{3\text{LSB}}$ of the second swingbys, based on the condition of the first swingby; (b) the upper limits of the $C_{3\text{LSB}}$ achieved by short swingbys, double swingbys and multiple swingbys. The upper limit is a quick assessment of the subsequent swingbys for Earth escape without numerically solving the Moon-to-Moon transfers, indicating the worth of performing subsequent swingbys. The planar Moon-to-Moon transfers are solved for the four manifold-guided encounters acquired in Sec. 3.2.

![Fig. 5. The $C_{3\text{LSB}}$ magnitudes and escape directions for the manifold-guided lunar swingbys and the direct escape along the anti-Sun-ward unstable manifold. ($A_i = 4 \times 10^3$ km)](image)

3.1. Graphical analysis of the lunar swingby Earth escape capacity varied by the solar tidal force

Given a set of $v_\infty$ and $\phi$ between the $v_\infty$ and $v_M$, one can compute the variables at the lunar swingby, such as velocity $v$, osculating radius of apogee $r_a$ as well as specific energy $\varepsilon$, and encounter angle $\alpha$ between $v$ and $v_M$. In addition, the three-body energy Jacobi integral $J$ in the CR3BP can be computed based on $v$ and an arbitrary position in Moon’s orbit. Although the $J$ would vary with the position along Moon’s orbit and the velocity direction, it only varies within a negligible range. Note that these variables can refer to pre- or post-swingby states by using superscripts “-” or “+”. Recalling Eq. (3)-4), $C_{3\text{LSB}}$ is determined by $v_\infty$ and angle $\phi$, i.e. $S$. Variations of these variables are plotted on the $v_\infty$-$\phi$ plane (Fig. 7). A close-up view of Fig. 7 is shown in Fig. 8 for explaining the lunisolar gravitational influence on $C_{3\text{LSB}}$. By a swingby, $v_\infty$ remains constant while $\phi$ changes to $\phi'$. To illustrate, in Fig. 8 a pre-swingby state $a_{S1}$ jumps vertically to different $\phi'$ levels as well as $J'$ levels by a swingby. The contour of $r_a = 8 \times 10^3$ km is drawn to separate the regions where the solar perturbation is
insignificant (up) and significant (down). If the post-swingby state is below this contour, such as \( a_{s1+} \) in the figure, the subsequent \( S1 \)-to-\( S2 \) transfer is perturbed by the Sun but \( J \) is maintained (see Appendix A). Therefore, the state will slide along the \( J \) contour during the \( S1 \)-to-\( S2 \) transfer, and terminate at an \( S2' \), such as \( a_{s2+} \) and \( a_{s2-} \). On the contrary, if the post-swingby state is above the \( r_\infty = 8 \times 10^3 \) km contour, such as \( a_{s1-} \), which suggests a nearly unperturbed transfer, the \( v_\infty \) will not change during the Moon-to-Moon transfer, thus \( S2 = S1' \). The \( S2' \) state reflects the \( C_{3LSB}^{\text{max}} \) of the second swingby. In summary, the graph can represent the influence of Moon's (vertical motion) and Sun's (motion along \( J \) contours) gravity on \( C_{3LSB}^{\text{max}} \).

For unperturbed transfers, no matter how many times of lunar swingbys are performed, as \( v_\infty \) is invariant, the variation of \( C_{3LSB}^{\text{max}} \) is confined by vertical motions on the graph. The upper limit of the \( C_{3LSB}^{\text{max}} \) for various \( v_\infty \) is found to be 2.3 km/s\(^2\) at the tangent point of the contours of \( r_\infty = 8 \times 10^3 \) km and \( C_{3LSB}^{\text{max}} \) at \( v_\infty = 1.35 \text{ km/s} \) (\( g_{3LSB} \)). To further increase \( C_{3LSB}^{\text{max}} \), the region affected by solar perturbation should be explored.

Inspecting Fig. 7, the contours of \( J \) are shown tangential to the contours of \( C_{3LSB}^{\text{max}} \) at \( \alpha \approx 80^\circ \). High \( \alpha \) results in high \( v_\infty \), but limited bending angle. \( \alpha \approx 80^\circ \) appears to be the optimal encounter condition for a post-swingby trajectory to get most \( C_{3LSB}^{\text{max}} \). If the new \( S2' \) state approaches \( \alpha = 80^\circ \), such as to \( a_{s2+} \), the \( C_{3LSB}^{\text{max}} \) will be increased. Oppositely, if the state slides away from \( \alpha = 80^\circ \), such as to \( a_{s2-} \), the \( C_{3LSB}^{\text{max}} \) will be decreased. In addition, along the \( \alpha = 80^\circ \) contour, \( C_{3LSB}^{\text{max}} \) increases with \( J \). In conclusion, there are two directions to maximize the \( C_{3LSB}^{\text{max}} \) at \( S1 \), one is increasing \( J \) by reducing \( \phi^+ \) at \( S1 \), another is altering the \( \alpha \) at the end of the \( S1 \)-to-\( S2 \) transfer to \( 80^\circ \). Then, one can estimate the upper limit of the \( C_{3LSB}^{\text{max}} \) of the \( S2 \) based on the \( S1 \) state. Considering the manifold-guided \( S1 \), for type 1 and type 2, the \( V_\infty \) ranges from 0.24 to 0.6 km/s for the considered range of \( A_t \) (see Fig. 2). Points \( b_{1f} \), and \( c_{1f} \) locate \( \phi^+ = 0^\circ \) as well as maximum \( J^+ \) for the two \( V_\infty \) boundaries. For \( c_{1f} \), along the \( J \) contours linking with \( b_{1f} \) and \( c_{1f} \), the upper limit of the \( C_{3LSB}^{\text{max}} \) are found to be 2.6 and 3.7 km/s\(^2\) at \( \alpha = 80^\circ \) (\( b_{1f} \) and \( c_{2f} \)). For type 3 and type 4, the \( V_\infty \) are around 1.35 km/s and \( \phi \) are 120°. However, the \( v_\infty \) magnitude constrains the minimum \( \phi^+ \). It can be computed that, \( \phi^+ > 50^\circ \). Then, the upper limit of \( C_{3LSB}^{\text{max}} \) along the \( J \) contour that links with the state \( e_{1f} \) (1.35, 50) is 5.6 km/s\(^2\) (\( e_{2f} \)). The upper limit of the \( C_{3LSB}^{\text{max}} \) of the \( S2 \) for the four types of \( S1 \) are shown as a function of \( A_t \) in Fig. 9. The limitation of the graph should be noted. The lunar phase of \( S1 \), which defines the initial condition along with \( v_\infty \) and \( \phi \), and flight time are not specified in the graph. Not every \( S2' \) state along the \( J \) contour can be explored from the \( S1' \) state within a limited flight time. Nevertheless, given no flight time constraint, it is reasonable to assume that, every state the \( J \) contour can be achieved. Therefore, the upper limits acquired from the graph are the theoretical maxima that can be expected but should not be exceeded in the real world.

Since flight time is critical in space missions, it is limited to 200 days for further discussions. If the spacecraft departs to a hyperbolic orbit (\( \varepsilon > 0 \)) at \( S1 \), it will probably escape. In that case, orbits such as resonance orbits with the Earth, and Sun-Earth Lyapunov orbits that are large enough to intersect with Moon's orbit, are possible Moon-to-Moon transfer solutions. But these orbits require significant flight time generally longer than 200 days. With the flight time constraint, the \( S1 \)-to-\( S2 \) transfers are considered as perturbed elliptic transfers. Then, the change of \( C_{3LSB}^{\text{max}} \) is bounded by the contour with \( \varepsilon = 0 \). Hence, the limit of the \( C_{3LSB}^{\text{max}} \) of a second
lunar swingby would be around 3.3 km²/s², which is acquired at the tangent point of the contours of \( \epsilon = 0 \) and \( C_{3LSB}^{\text{max}} \left( f_{32} \right) \). In addition, following an elliptic transfer, if the swingby delivers the spacecraft to a retrograde orbit, which enables the spacecraft to encounter the Moon once more on the outbound leg, as is demonstrated in Ref. [5] and Ref. [12]. The solution sets of \( \varphi^* \) and \( v_r \) are acquired by solving the Lambert problem for varying positions along Moon’s orbit and the corresponding flight time, as represented by the green line in Fig. 10. As shown in the figure, the last second swingby can shift a state \( S2 \) to an \( S2^* \) along the green line upon \( \epsilon = 0 \), suggesting a hyperbolic transfer. As the consequent \( S2 \)-to-\( S3 \) transfer is within Moon’s orbit, the solar perturbation is not effective in changing the encounter state. Hence, \( S3 = S2^* \). As it is shown, states along the green line upon \( \epsilon = 0 \) suggest a hyperbolic transfers, which open up a pathway to increase \( C_{3LSB}^{\text{max}} \). However, the bending angle limit would constrain the energy of a hyperbolic transfer. The upper limit of the \( C_{3LSB}^{\text{max}} \) of a third swingby is found to be 5 km²/s² at the intersection of the lines of the transfer solution and lower boundary of \( \varphi^* \). Note that this upper limit also applies to the multiple lunar swingby case. As noted earlier, if the flight time is limited, not every state that can be connected with the \( S1 \) state on the graph is achievable. In other words, whether the upper limit of \( C_{3LSB}^{\text{max}} \) can be achieved within the limited flight time depends on the initial condition of \( S1 \).

Fig. 10. Acquiring the \( C_{3LSB}^{\text{max}} \) of a multiple lunar swingby.

### 3.2. Results and discussions of second lunar swingby

According to Fig. 9, a second lunar swingby might increase the \( C_{3LSB} \) to a considerable level of 3.3 km²/s² in 200 days. The second swingby transfers for the manifold-guided \( S1 \) are to be solved. The routine of solving the planar \( S1 \)-to-\( S2 \) transfers for a given \( S1 \) condition is based on variable step-size search and differential correction as explained in Appendix B.

The \( S1 \)-to-\( S2 \) transfers solved for each type of manifold-guided encounter for \( A_e = 4 \times 10^5 \) km are presented in this section. Because the problem of interest is high-\( C_3 \) Earth escape, the solutions with \( v_e \) lower than 0.5 km/s at \( S2 \) are not presented. Fig. 11 shows the transfers with one Sun-perturbed loop. Because the shown transfers result in increased \( v_e \) at \( S2 \) for type 1 and type 2, the trajectories are decelerated by the Sun. The apogees of the transfers for type 1 and type 2 are shown generally in the 1\( ^{\text{st}} \) or 3\( ^{\text{rd}} \) quadrants accordingly. Fig. 12 shows the two-loop transfers for type 2 as an example, with one solution highlighted. It can be seen that after one loop the apogee in the rotating frame shifts to the 1\( ^{\text{st}} \) or 3\( ^{\text{rd}} \) quadrants. Within 200 days, the solutions are found to be up to three loops for the \( S1 \) of type 3 and type 4 and two loops for the \( S1 \) of type 1 and type 2.

The \( C_{3LSB}^{\text{max}} \) of the \( S2 \) vs ToF is plotted in Fig. 13. The maximum of the \( C_{3LSB}^{\text{max}} \) of \( S2 \) found in 200 days is 2.7 km²/s² for all types. The \( S2 \) for the \( S1 \) of type 1 and type 2 achieve the practical maximum level of \( C_{3LSB}^{\text{max}} \) in 95 and 75 days, respectively. However, the second swingbys for the \( S1 \) of type 3 and type 4 cannot further increase the \( C_{3LSB}^{\text{max}} \) from the level of the first swingbys. The second swingby options with \( C_{3LSB}^{\text{max}} > 2 \) km²/s² steadily occur after 70 days.

As discussed earlier, \( J \) increase at \( S1 \) and \( \alpha \) approaching to 80° at \( S2 \) contribute to the increase of \( C_{3LSB}^{\text{max}} \). The \( S1 \) and \( S2 \) states are plotted in the \( \alpha-J \) plane in Fig. 14. It can be seen that there are some \( S2 \) states with \( \alpha \) near 80°, but rare states on the upper side where \( J \) and \( \epsilon \) are high. Because high-\( \epsilon \) orbits have long periods, within a finite ToF, the chance of re-encounter is rare for high \( \epsilon \). The upper limit of 3.3 km²/s² for double swingbys is given at \( \epsilon = 0 \) without specifying the condition of \( S1 \) (i.e. lunar phase, \( v_r \) and \( \varphi \)). A small distribution of \( S1 \) conditions can lead to transfers with \( \epsilon \approx 0 \) within 200 days. Since there are only four specific \( S1 \) conditions confined by using manifold transfer, the practical maximum (2.7 km²/s²) are shown not close to the theoretical limit. On the other hand, Ref. [12] have computed double lunar swingbys for various \( S1 \) conditions within 7 months (≈ 200 days) and shown that the maximum of \( C_{3LSB} \) is 3.25 km²/s². In the experiment of extending ToF (results of which are not presented in this paper), an \( S2 \) solution with \( C_{3LSB}^{\text{max}} \) of 3.23 km²/s² can be found for the \( S1 \) of type 2 in around 300 days, which is close to the limit of 3.3 km²/s² (see Fig. 9 for the value of 2\( ^{\text{nd}} \) swingby for type 2 at \( A_e = 4 \times 10^5 \) km). In another experiment with \( A_e = 1 \times 10^5 \) km, an \( S2 \) solution with \( C_{3LSB}^{\text{max}} \) of 2.6 km²/s² can be found for the \( S1 \) of type 1 in 70 days, which is consistent with the theoretical limit (see Fig. 9). Because the \( C_{3LSB}^{\text{max}} \) level of 2.6 km²/s² can be accomplished by transfers with moderate \( \epsilon \) as well as short flight time, which is likely to occur for any condition of \( S1 \).

In addition, it is interesting to compare the upper limits of \( C_{3LSB}^{\text{max}} \) acquired from the graphical analysis with the results in some literature. The summary of the comparison is:

i) For short swingbys with unperturbed transfers, the upper limit of the \( C_{3LSB}^{\text{max}} \) of 2.3 km²/s² is consistent with the maximum \( C_{3LSB} \) presented in Ref. [8] and Ref. [12] which compute various short double lunar swingbys.
ii) For double swingbys with perturbed transfers, it has been mentioned that within 200 days the upper limit of the $C_{3LSB}^{\max}$ of $3.3 \text{ km}^2/\text{s}^2$ is consistent with Ref. [12].

iii) For multiple swingbys with perturbed transfers, within 200 days, the upper limit of the $C_{3LSB}^{\max}$ of $5 \text{ km}^2/\text{s}^2$ is also consistent with Ref. [12], which shows that the maximum of the $C_{3LSB}$ achieved by various triple swingby is $4.7 \text{ km}^2/\text{s}^2$.

With many S2 solutions derived from the S1 conditions, the escape directions should be further broadened. Again, the post-swingby states are propagated to the SoE. The escape directions and the maximum $C_{3SoE}$ in these directions are printed in Fig. 15. A $C_{3SoE}$ greater than 2 $\text{km}^2/\text{s}^2$ is generally available at all longitudes. In an SEP mission, the SEP can be used to steer the spacecraft back to Earth for gravity assist and meanwhile change the $v_{\infty}$ with respect to the Earth for an Earth gravity assist. The technique is called as the Electric Propulsion Delta-V Earth Gravity Assist (EDVEGA).[13] Ref. [8] has shown that an escape $C_3$ of $2 \text{ km}^2/\text{s}^2$ can be well used and greatly increased by the EDVEGA with a modern SEP, enabling missions to targets as far as the Jupiter. Therefore, the $C_3$ achieved by the concerned escape strategy would be significant in establishing a deep space mission.
for the S1 of type 2 with one solution highlighted (Earth-centered rotating frame). S1 are marked by circles and S2 dots. \( A_e = 4 \times 10^5 \) km

![Figure 13. \( C_{3LSB} \) of the second swingbys vs ToF.](image)

Motivated by the increasing interests in halo orbit missions, the Earth escape strategy of utilizing the unstable manifold and lunar gravity assists is proposed and analyzed. First, the four types of manifold-guided lunar encounters are acquired. Two types of them can lead to effective lunar gravity assists and a maximum \( C_{3LSB} \) of 2.6 \( \text{km}^2/\text{s}^2 \) with respect to the Earth.

As the \( v_\infty \) of the encounters of type 1 and type 2 are not high for the lunar swingbys to effectively increase \( C_{3LSB} \), subsequent lunar swingbys utilizing solar tidal force to change the \( v_\infty \) and thus the maximum achievable \( C_{3LSB} \) are discussed. A graphical analysis is presented in this paper. Without numerically solving various Moon-to-Moon transfers, it reveals that the theoretical upper limits of the \( C_{3LSB} \) achieved by short swingbys without solar perturbed transfer, double and multiple swingbys are 2.3 \( \text{km}^2/\text{s}^2 \), 3.3 \( \text{km}^2/\text{s}^2 \) and 5 \( \text{km}^2/\text{s}^2 \), respectively. With a ToF = 200 days, the practical second swingby solutions for the four types of manifold-guided encounter can achieve a maximum \( C_{3LSB} \) of 2.7 \( \text{km}^2/\text{s}^2 \). In particular, for type 1 and type 2, allowing another 75-95 days to perform a second swingby, the \( C_{3LSB} \) can be increased to the practical maximum. Therefore, for the S1 with low \( v_\infty \), \( C_{3LSB} \) can be efficiently increased by applying a second lunar swingby. Comparing with the direct manifold escape, the lunar swingby option may take longer flight time to achieve Earth escape. However, the swingby options that lead to considerable \( C_3 \) largely broaden the choices for a direct visit to heliocentric bodies as well as enhance the EDVEGA technique, to which attention should be paid when the extended mission for halo orbits is considered.

The study assumes the halo orbit mission has been pre-phased for a lunar encounter. However, for general extended missions, the halo orbit mission should not be pre-phased for a future destination. Because the use of unstable manifolds along with lunar gravity assists is shown advantageous, a follow-up work will concern the phasing cost.

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**Appendix A**

The **Circular Restricted Three-Body Problem** (CR3BP) assumes two primary bodies moving in a circular orbit about their barycenter. The mass of the third body is negligible compared to the masses of the two primaries. The two primaries can be the Sun and Earth, the Earth and Moon or etc. The rotating coordinate system with the barycenter at the origin and the primaries fixed on the x-axis is chosen to describe the motion of the third body.
For convenience, the angular velocity of the rotating frame, the total mass and the distance between the two primaries are normalized to 1. \( \mu \) is the ratio of the mass of secondary primary body \( m_2 \) to mass of the first primary body \( m_1 \). The \( m_1 \) and \( m_2 \) become 1-\( \mu \) and \( \mu \) in the dynamical model. Then, the coordinates of the first and secondary bodies are \([-\mu, 0, 0]\) and \([1-\mu, 0, 0]\) respectively, as shown in Fig. A.1. The equations of motion of the third body are:

\[
\begin{align*}
\dot{x} - 2y &= \frac{\partial U}{\partial x} \\
\dot{y} + 2x &= \frac{\partial U}{\partial y} \\
\dot{z} &= \frac{\partial U}{\partial z}
\end{align*}
\]

where the pseudo-gravitational potential \( U \) in the system is

\[
U = \frac{(x^2 + y^2)}{2} + (1-\mu)(r_1 + \mu)/r_2
\]

where

\[
r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}
\]

The system has an energy integral,

\[
J = 2U - \left( x^2 + y^2 + z^2 \right)
\]

which is a conserved quantity called Jacobi integral.

**Equilibrium points**

Ref. [14] has presented the detail of resolving the equilibrium points from the equations of motion. They are also referred to as Lagrangian points labeled as \( L_1, \ldots, L_5 \). The three collinear points \( L_1, L_2 \) and \( L_3 \), are unstable. The geometries of the five equilibrium points are depicted in Fig. A.1.

![Fig. A.1. Circular restricted three-body system and Lagrangian points](image)

**Appendix B**

The position and \( v_x^* \) of S1 are associated with the size of the halo orbit and the encounter type. Therefore, the initial condition of a S1-to-S2 transfer is defined by the direction of \( v_x^* \). In the planer case, the angle \( \beta \) with respect to the Earth-Moon axis is used to specify the direction of \( v_x^* \). After a time of flight (ToF), the distance to Moon’s orbit \( \Delta d \) and phase difference \( \Delta \theta \) from the lunar phase should be zero for a re-encounter event. The \( \beta \) and ToF determine whether the spacecraft re-encounters the Moon. Namely, the solution set is,

\[
\{ (\beta, ToF) | \Delta d = 0, \Delta \theta = 0 \}
\]

In searching the solutions, \( \beta \) is changed from 0° to 360° in an initial increment of 0.05°. The initial conditions that cannot have osculating \( r_v \) greater than \( 8 \times 10^5 \) km are excluded from the search. Otherwise, there will be solutions of resonances with the Moon, which are not the major concern here and can be solved in a simpler way. The initial state of given \( \beta \) is propagated until the trajectory reaches the Moon’s orbit from outside (inbound case). Thus, at the termination, the trajectory has passed through the apoapse where the solar perturbation exerts the greatest influence on the trajectory. Moreover, this step imposes \( \Delta d = 0 \) and can return the corresponding ToF and \( \Delta \theta \). Then, the problem is simplified to locating \( \Delta \theta = 0 \). If the sign of \( \Delta \theta \) changes at two consecutive samples, the interval between the two samples should include a solution. Close guesses of solutions are obtained by interpolating \( \beta \) and ToF at \( \Delta \theta = 0 \) in the solution-including intervals. The last step is to get the accurate \( \beta \) and ToF by performing differential correction to target Moon’s position. Note that the propagation time is limited to 200 days. Solutions in the \( \beta \) intervals that cannot lead to a comeback to Moon’s orbit in 200 days are excluded from discussion. In addition, the \( \beta \) intervals that are not inside the bendable angle domain are also excluded.

On the other hand, \( \Delta \theta \) becomes very sensitive to \( \beta \) as the period of the post-swingby orbit increases. A tiny change of \( \beta \) can result in a great shift of \( \Delta \theta \). Consequently the true \( \Delta \theta \) variation cannot be captured by the fixed step size sampling. To cope with this, the step size of \( \beta \) is adjusted to maintain the changes of \( \Delta \theta \) not greater than 30°. The resolution of \( \beta \) is down to 0.007° for the longest ToF cases, and the increment as great as 1.5° is found sufficient for the shortest ToF cases. With the step size control, the algorithm becomes efficient and unlikely to miss solutions. In addition, there is another case of lunar encounter. The one passes Moon’s orbit on inbound leg is already considered. The other one passes Moon’s orbit from inside (outbound case). The outbound case can be easily derived from the states at the inbound crossing as the trajectory has come to the near-Earth realm where the Keplerian elements can be applied to acquire \( \Delta \theta \) and ToF at the outbound crossing. Moreover, there are also cases that the trajectory passes inbound Moon’s orbit several times before a re-encounter. To find multiple-loop solutions, the program is modified to identify all inbound crossings and return the corresponding \( \Delta \theta \) and ToF variations for solution search. The program is executed by Matlab R2012a on the machine with CPU: Intel i7-3770 @ 3.4 GHz and RAM: 8GB. For given position and \( v_x^* \) of S1, the program generally finds all Sun-perturbed Moon-to-Moon solutions in 2 minutes.

**References**


