# The Design of PID Control by Coefficient Diagram Method

**Shunji Manabe**

*Retired (Formerly Mitsubishi Electric and Tokai University)*
*1-8-12 Kataseyama, Fujisawa-shi, Kanagawa 251-0033, Japan*
*(e-mail:manabes@cityfujisawa.ne.jp)*

**Astract:** The PID control is widely used in various control systems. For the selection of controller parameters, the Ziegler-Nichols approach is known at present. However it is usually applicable to multistage time-lag plants, and not to the oscillatory plants. The CDM (Coefficient Diagram Method) is applicable to any kind of plants. In this paper, CDM is applied to the controller design of the multistage time-lag plants, and the results are compared with the Ziegler-Nichols approach.

**Keywords:** Control system design, Control theory, PID control, Polynomial approach.

# 係数図法による PID 制御系の設計

**内容梗概:** PID 制御系は現在広く応用されていて、制御器パラメータの選定法には Ziegler-Nichols の方法が現在知られている。しかしこの方法では制御対象として、多段時定数回路を想定しており、振動系などには適用できない。係数図法はどのような制御対象の制御器の設計にも適用できる。本報告は PID 制御が対象としている、多段時定数回路的な制御対象について、係数図法の設計を行い、Ziegler-Nichols の方法と比較している

## 1. INTRODUCTION

The PID control is widely used in various control systems. The Ziegler-Nichols approach is known at present for the controller parameter design, as shown in various control textbooks such as the ones by Franklin (1994, p.191), Åström (2008, p.302), or Qiu (2010, p.299, p.304). The history of PID control is well described by Babb (1990) and Brickery (1990). The PID control was developed mainly for the purpose of controlling multistage time-lag plants. Thus it is not applicable to other plants such as oscillatory plants. The CDM (Coefficient Diagram Method) is developed for the control of all kinds of plants, including multistage time-lag plants and oscillatory plants (Manabe, 1998, 2002, 2012). In order to clarify the nature of PID control, CDM is applied to the design of multistage time-lag plants and the results are compared with the results obtained by the Ziegler-Nichols approach.

The paper is organized as follows. The usual multistage time-lag plants are represented by a plant consisting of a pure time delay and a time-lag. Section 2 discusses the way of approximating the pure time-delay by polynomials such that CDM design is applicable, because only polynomials are used in CDM. In section 3, the method of representing a discrete system by a continuous system with a pure time-delay is discussed. Thus the design of a discrete PID controller can be made by the small addition of such a pure time-delay to the original plant. In section 4, the CDM designs are made for the plant consisting of a pure time-delay $L$ and a time-lag $T$ for various $L/T$ ratios, especially for $1/T=\infty$ or $L/T=\infty$. In section 5, these design results are compared with results by the Ziegler-Nichols approach. In section 6, the results are summarized in conclusions.

## 2. REPRESENTATION OF TIME DELAY

In order to approximate the pure time-delay, proper consideration is necessary. First, the approximation is to be easy to handle in CDM design. For this reason, the denominator-polynomial type is chosen. Second, the time-delay, $e^{-Ls}$, is to be approximated accurately only up-to $\omega=1/L\ rad/\sec$, which corresponds to 57.296 degrees phase shift. This condition is introduced from the finding that the usual closed system with time-delay must have an integrator with 90 degrees phase shift and also about 40 degrees phase margin for stability. Thus the contribution of the phase shift by the time-delay is limited to 50 degrees at the maximum. Third, under this environment, the stability condition must be sufficiently accurate. Forth, the step response of the closed-loop must be accurate in practical sense.

To satisfy the first condition, the following approximation is suggested.

$$e^{-Ls} \approx \frac{1}{A_L(s)} = \frac{1}{0.1L^3s^3 + 0.5L^2s^2 + Ls + 1}. \quad (1)$$

The second condition is verified by the comparison of the frequency responses at $\omega = 1/L\ rad/\sec$.

$$e^{-Ls} = 1\angle -57.296\ \text{deg}, \qquad 1/A_L(s) = 0.97129\angle -60.945\ \text{deg}. \quad (2)$$

In order to verify the third condition, the stability conditions are compared. For this purpose, the system consisting of a time-delay and an integrator is considered. This system is most stringent in terms of verifying the approximation, because the effect of the time-delay is largest with about 50 degrees phase shift. The open-loop transfer function of the system is shown as follows.

$$G(s) = e^{-Ls}K/s \approx K/(sA_L(s)). \quad (3)$$

The stability condition and the oscillation frequency $\omega$ for the exact time-delay system are obtained by making the loop gain to 1, while the phase is made as $-\pi\ \text{rad} = -180\ \text{deg}$ for $s = j\omega$.

$$K = 1.5708/L, \quad \omega = \pi/(2L) = 1.5708/L. \quad (4)$$

The stability condition for the approximated system can be obtained from the characteristic polynomial $P(s)$ of the closed-loop system and the condition that $P(j\omega) = 0$.

$$\begin{aligned} P(s) &= 0.1L^3s^4 + 0.5L^2s^3 + Ls^2 + s + K, \\ &= a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0, \quad (5) \\ a_2 &= a_4(a_1/a_3) + a_0(a_3/a_1), \quad \omega = (a_1/a_3)^{0.5}. \end{aligned}$$

This leads to the following results.

$$K = 1.6/L, \quad \omega = 1.4142/L. \quad (6)$$

The comparison of the above results shows that the third condition is met.

The system to verify the forth condition consists of a time-delay and an integrator with $K = 0.4/L$. The close-loop response of the exact system $W(s)$ is known to have no overshoot (Manabe, S., 2003).

$$W(s) = G(s)/(1+G(s)) = 0.4/(Lse^{Ls} + 0.4). \quad (7)$$

For the approximation, $W(s)$ is as follows.

$$W(s) = 0.4/(0.1L^4s^4 + 0.5L^3s^3 + L^2s^2 + Ls + 0.4). \quad (8)$$

From the stability indexes, $[\gamma_3\ \gamma_2\ \gamma_1] = [2.5\ \ 2\ \ 2.5]$, it is clear that this response has no overshoot.

The responses for $L = 1$ are shown in Fig. 1. They are almost equal.

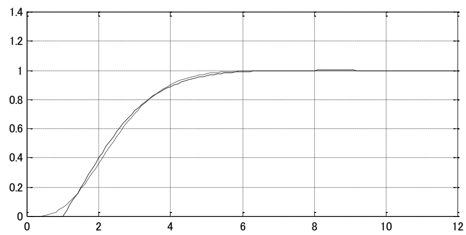


Fig. 1. Comparison of responses, —— exact, ---- approximate

## 3. REPRESENTATION OF SAMPLED-DATA SYSTEM

In ordinary PID control, the controller is the discrete sampled-data system, while the plant is the continuous system. In order to analyze the total system, the discrete sampled-data controller has to be approximated by the continuous system with some equivalent time-delay. Such equivalent time-delay is about 0.5~1.5 times of the sampling period $T_s$, depending on the input function and controller function, as shown in the following three examples.

The first example is the case, when the controller transfer function is unity, $G_c(s) = 1$, and the input function is a ramp function, $x(t) = t$. In this case the equivalent time-delay is $1.5T_s$, as shown in Fig. 2. The relation is shown in the following equation.

$$y_{out}(t) = t - 1.5T_s = x(t - 1.5T_s) = e^{-1.5T_s s}x(t). \quad (9)$$

The analog input $x(t) = t$ is converted to digital signal and stored as a digital data. The data is processed according to the specified transfer function and converted to analog signal at the next sampling time. The analog signal is held until to the next sampling time by the sample-hold devise, as $y(t) = x(t - T_s)$. The equivalent analog output is further delayed by $0.5T_s$ as the average of the step-wise $y(t)$ with the result $y_{out}(t) = e^{-1.5T_s s}x(t)$.

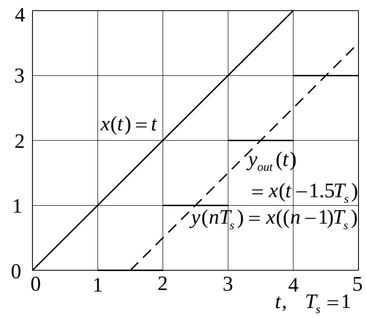


Fig. 2. Input-output relation, $G_c(s) = 1$

The second example is the case, when $G_c(s) = 1/s$ with Euler integration formulae and the input function is a ramp function, $x(t) = t$. In this case the equivalent time-delay is $T_s$. The Euler integration formulae is as follows.

$$y(nT_s) = y((n-1)T_s) + T_s x((n-1)T_s). \quad (10)$$

This equation can be rewritten as

$$\begin{aligned} &[y((n-0.5)T_s + 0.5T_s) - y((n-0.5)T_s - 0.5T_s)]/T_s \\ &\quad = x((n-0.5)T_s - 0.5T_s). \end{aligned} \quad (11)$$

The left hand term can be approximated as $sy((n-0.5)T_s)$. By replacement of $(n-0.5)T_s = t$, the next relation is obtained.

$$sy(t) = x(t - 0.5T_s). \quad (12)$$

The averaged output is further delayed by $0.5T_s$ and the final result is as follows.

$$y_{out}(t) = (1/s)e^{-T_s s}x(t). \quad (13)$$

The third example is the case when the advancing modified

Euler integration formula is used in the second example. In this case the equivalent time-delay is $0.5T_s$. The advancing modified Euler integration formula is as follows.

$$\begin{aligned} y(nT_s) &= y((n-1)T_s) \\ &\quad + T_s[1.5x((n-1)T_s) - 0.5x((n-2)T_s)]. \end{aligned} \tag{14}$$

This equation can be rewritten as

$$\begin{aligned} &[y((n-0.5)T_s + 0.5T_s) - y((n-0.5)T_s - 0.5T_s)]/T_s \\ &= 1.5x((n-1)T_s) - 0.5x((n-2)T_s) \\ &= x((n-0.5)T_s - 0.5T_s) \\ &\quad + 0.5[x((n-1)T_s - x((n-2)T_s)]. \end{aligned} \tag{15}$$

The second term of the right hand side can be further modified as follows.

$$\begin{aligned} &0.5[x((n-1)T_s - x((n-2)T_s)] \\ &= 0.5T_s[x((n-1.5+0.5)T_s) - x((n-1.5-0.5)T_s)]/T_s \\ &\approx 0.5T_s sx((n-1.5)T_s). \end{aligned} \tag{16}$$

Then Eq. (16) becomes, by the use of $(n+0.5)T_s = t$, as follows.

$$\begin{aligned} y(t) &= (1/s)[x(t-0.5T_s) + 0.5T_s sx(t-T_s)] \\ &= (1/s)e^{-0.5T_s s}[1 + 0.5T_s se^{-0.5T_s s}]x(t). \end{aligned} \tag{17}$$

For $x(t) = t$, $y(t)$ becomes $(1/s)x(t)$. Thus

$$y_{out}(t) = (1/s)e^{-0.5T_s s}x(t). \tag{18}$$

The system time delay is the sum of $(0.5 \sim 1.5)T_s$ and the original time delay $L$. Because of $T_s \ll L$, the effect of sampling can be usually neglected.

## 4. CDM DESIGN

### *4.1 Design Procedure*

The block diagram of the control system is shown in Fig. 3 in the standard CDM expression. The plant is given as follows.

$$\begin{aligned} G_p(s) &= \frac{B_p(s)}{A_p(s)} = e^{-Ls}\frac{K}{Ts+1} \\ &\approx \frac{K}{A_L(s)(Ts+1)} = \frac{R}{A_L(s)(s+1/T)}, \\ A_L(s) &= 0.1L^3s^3 + 0.5L^2s^2 + Ls + 1, \quad R = K/T, \\ A_p(s) &= 0.1L^3s^4 + 0.5L^2(1+0.2L/T)s^3 \\ &\quad + L(1+0.5L/T)s^2 + (1+L/T)s + 1/T, \\ B_p(s) &= R. \end{aligned} \tag{19}$$

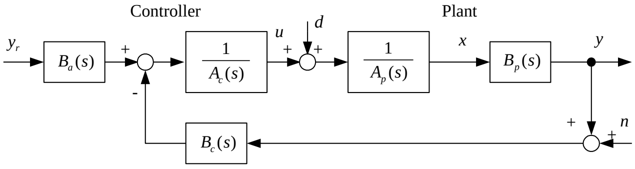


Fig. 3. Block diagram of control system in CDM

The controller is assumed as a PI controller. This assumption will be validated by the design results.

$$G_c(s) = \frac{B_c(s)}{A_c(s)} = \frac{k_1s + k_0}{s}. \tag{20}$$

The characteristic polynomial is obtained as follows.

$$\begin{aligned} P(s) &= A_c(s)A_p(s) + B_c(s)B_p(s) = \sum_{i=0}^{5} a_is^i, \\ a_5 &= 0.1L^3, \quad a_4 = 0.5L^2(1+0.2L/T), \\ a_3 &= L(1+0.5L/T), \quad a_2 = 1 + L/T, \\ a_1 &= 1/T + Rk_1, \quad a_0 = Rk_0. \end{aligned} \tag{21}$$

Design is made for $\gamma_2 = 2, \gamma_1 = 2.5$. The second order equivalent time constant $\tau_2$ is obtained from the plant parameters. Thus the equivalent time constant is obtained as follows.

$$\begin{aligned} \tau_2 &= \frac{a_3}{a_2} = \frac{L(1+05L/T)}{1+L/T}, \quad \tau_1 = \gamma_2\tau_2, \\ \tau &= \gamma_2\gamma_1\tau_2 = \frac{5L(1+0.5L/T)}{1+L/T}. \end{aligned} \tag{22}$$

For $L/T = 0$, values of $\tau_2, \tau_1,$ and $\tau$ are $L, 2L,$ and $5L$ respectively. Controller parameters are obtained in the following manner.

$$\begin{aligned} a_1 &= \frac{a_2}{\tau_1} = \frac{0.5(1+L/T)^2}{L(1+0.5L/T)}, \\ a_0 &= \frac{a_1}{\tau} = \frac{0.1(1+L/T)^3}{L^2(1+0.5L/T)^2}, \\ k_1 &= \frac{a_1 - 1/T}{R} = \frac{0.5}{RL(1+0.5L/T)}, \\ k_0 &= \frac{a_0}{R} = \frac{0.1(1+L/T)^3}{RL^2(1+0.5L/T)^2}. \end{aligned} \tag{23}$$

In usual cases, $L/T = 0$ is satisfied. Then the above results become as follows..

$$\begin{aligned} a_1 &= 0.5/L, \quad a_0 = 0.1/L^2, \\ k_1 &= 0.5/(RL), \quad k_0 = 0.1/(RL^2). \end{aligned} \tag{24}$$

For these design, the hand calculation can be systematically carried out by filling the CDM form as shown in Fig. 4, when plant parameters are given in concrete numbers.

### *4.2 Design Results*

Design can be made by more simpler manner with the use of gc command of CDM-CAD (Manabe, 2012), where controller parameters are automatically calculated when the equivalent time constant $\tau$ is given. The command line, for $R = L = 1, 1/T = 0$, is as follows.

```
RR=1;L=1;Tinv=0;aL=[0.1*L^3 0.5*L^2 1 1];
ap=[conv(aL,[1 Tinv]) 0];bp=RR;nc=0;mc=1;gr=[2 2 2 2.5]; t=5*L*(1+0.5*L*Tinv)/(1+L*Tinv);tm=0.5;gc
```

In the above, the integrator of the controller is moved to the plant. Thus the order of controller denominator $n_c$ is 0, while that of the numerator $m_c$ is 1. The gr is the reference stability indexes. Because the design freedom is two, $k_1$ and $k_0$, only $\gamma_1$ and $\gamma_2$ take the reference values, and other indexes, $\gamma_3$ and $\gamma_4$, have to be accepted as the results of design. The tm is the time scale of the step response, and usually takes the value 0.5 or 1.

The design results for $R = L = 1, 1/T = 0$ are as follows.

$$G(s) = G_c(s)G_p(s) = \frac{0.5s + 0.1}{s}\frac{1}{0.1s^4 + 0.5s^3 + s^2 + s},$$

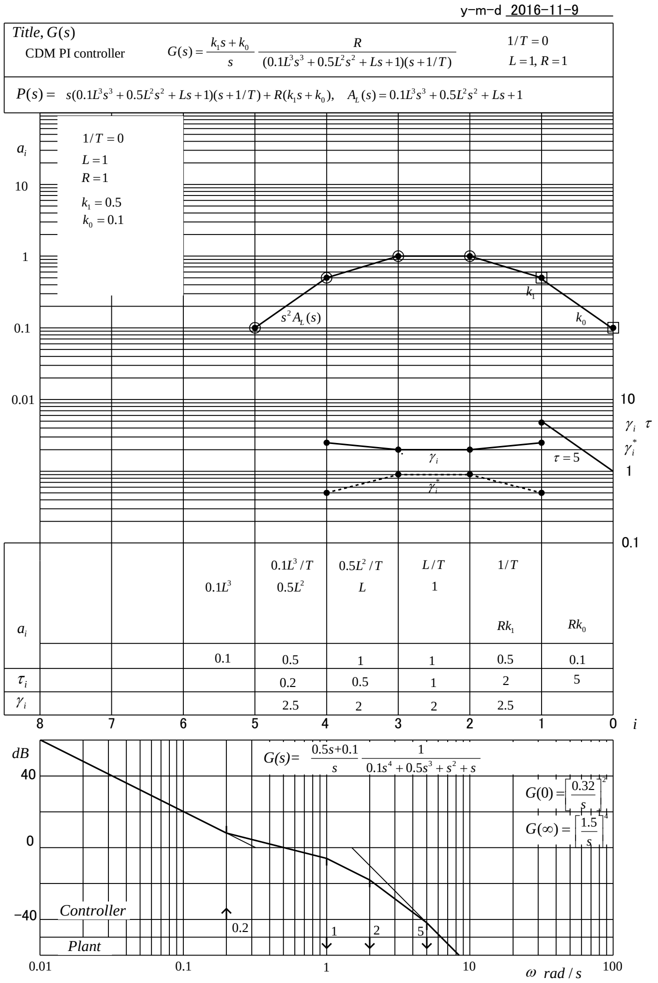


Fig. 4. Coefficient diagram, CDM, $R = L = 1, 1/T = 0$

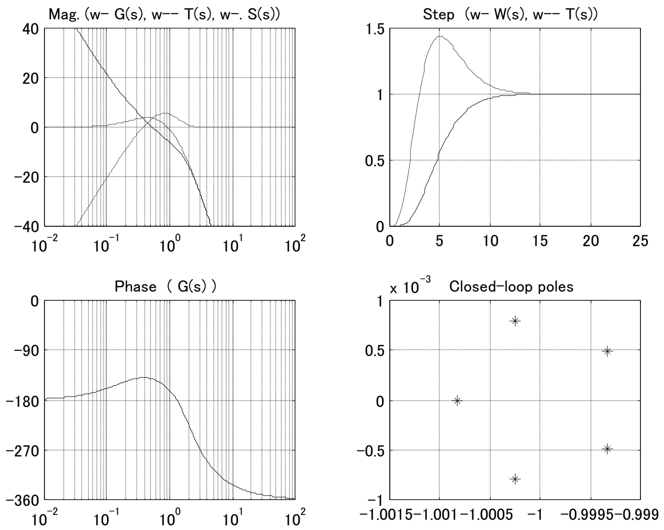


Fig. 5. Responses, CDM, $R = L = 1, 1/T = 0$

$$\begin{aligned} &P(s) = 0.1s^5 + 0.5s^4 + s^3 + s^2 + 0.5s + 0.1, \\ &\gamma_i = [2.5 \;\; 2 \;\; 2 \;\; 2.5], \quad \tau = 5, \\ &s_i = -1, -1, -1, -1, -1, \\ &\phi_m = 38.319 \text{ deg}, \quad g_m = 2.7778. \end{aligned} \tag{25}$$

The $s_i$, $\phi_m$, and $g_m$ are closed-loop poles, phase margin and gain margin respectively. The coefficient diagram with related Bode diagram is shown in Fig. 4. The responses are shown in Fig. 5. The $G(s), S(s),$ and $T(s)$ are open-loop transfer function, sensitivity function, and complementary sensitivity function respectively, and $W(s)$ is the transfer function for the command following characteristics, where the reference numerator $B_a(s)$ is $k_0$. From values of the high order stability indexes, $\gamma_4 = 2.5$ and $\gamma_3 = 2$, it will be concluded that PI controller suffices.

## 5. COMPARISON WITH ZIEGLER-NICHOLS APPROACHES

The CDM design results are obtained as in Eq. (23). For $L/T = 0$, they become as follows.

$$k_1 = 0.5/(RL), \quad k_0 = 0.1/(RL^2). \tag{26}$$

The above is the case for PI controller. For P controller, the results are as follows.

$$k_1 = 0.4/(RL), \quad k_0 = 0. \tag{27}$$

The results are obtained by making $\gamma_2 = 2.5$ and $\gamma_1 = \infty$ in Eqs. (22)(23)(24).

There are two methods in Ziegler-Nichols approaches. The first method, Quarter Decay Ratio (QDR), is based on the condition that the amplitude decay in one cycle of oscillation is about 0.25 in P control. The results are as follows.

$$\begin{aligned} &G_c(s) = K_P(1 + \frac{1}{T_I s}), \\ &K_P = 0.9/(RL), \quad T_I = L/0.3, \quad \text{for PI control}, \\ &K_P = 1/(RL), \quad T_I = \infty, \quad \text{for P control}. \end{aligned} \tag{28}$$

These values are converted to CDM parameters as follows.

$$\begin{aligned} &k_1 = K_P = 0.9/(RL), \\ &k_0 = K_P/T_I = 0.27/(RL^2), \quad \text{for PI control}, \\ &k_1 = 1/(RL), \quad k_0 = 0, \quad \text{for P control}. \end{aligned} \tag{29}$$

The second method, Marginally Stable System (MSS), is based on the stability limit at P control.

$$\begin{aligned} &K_P = 0.45K_U, \quad T_I = P_U/1.2, \quad \text{for PI control}, \\ &K_P = 0.5K_U, \quad T_I = \infty, \quad \text{for P control}. \end{aligned} \tag{30}$$

The characteristic polynomial for P control with $1/T = 0$ is given as follows.

$$P(s) = 0.1L^3s^4 + 0.5L^2s^3 + Ls^2 + s + RK_U. \tag{31}$$

The stability condition for fourth order polynomial is as follows.

$$a_2 = a_4(a_1/a_3) + a_0(a_3/a_1). \tag{32}$$

From this condition the following results are obtained.

$$K_U = 1.6/(RL), \quad P_U = 2\pi/\omega_U = 2\pi/\sqrt{a_1/a_3} = \sqrt{2}\pi L. \tag{33}$$

These values are converted to CDM parameters as follows.

$$k_1 = K_P = 0.45K_U = 0.72/(RL),$$
$$k_0 = K_P/T_I = \frac{0.72/(RL)}{\sqrt{2}\pi L/1.2} = 0.19447/(RL^2), \text{ for PI control,} \quad (34)$$
$$k_1 = 0.8/(RL), \quad k_0 = 0, \quad \text{for P control.}$$

QDR design results for $R = L = 1, 1/T = 0$ are as follows.

$$G(s) = G_c(s)G_p(s) = \frac{0.9s+0.27}{s}\frac{1}{0.1s^4+0.5s^3+s^2+s},$$
$$P(s) = 0.1s^5+0.5s^4+s^3+s^2+0.9s+0.27, \quad (35)$$
$$\gamma_i = [2.5\ \ 2\ \ 1.1111\ \ 3], \quad \tau = 3.3333,$$
$$s_i = -2.1205 \pm j1.0154, \ -0.16008 \pm j1.0427, \ -0.4389,$$
$$\phi_m = 16.046 \text{ deg}, \quad g_m = 1.4084.$$

The coefficient diagram is shown in Fig. 6. The responses are shown in Fig. 7.

MSS design results for $R = L = 1, 1/T = 0$ are as follows.

$$G(s) = G_c(s)G_p(s) = \frac{0.72s+0.19447}{s}\frac{1}{0.1s^4+0.5s^3+s^2+s},$$
$$P(s) = 0.1s^5+0.5s^4+s^3+s^2+0.72s+0.19447,$$
$$\gamma_i = [2.5\ \ 2\ \ 1.3889\ \ 2.6657], \quad \tau = 3.7024, \quad (36)$$
$$s_i = -1.9778 \pm j0.87761, \ -0.29624 \pm j0.91181, \ -0.45189,$$
$$\phi_m = 25.337 \text{ deg}, \quad g_m = 1.8125.$$

The coefficient diagram is shown in Fig. 8. The responses are shown in Fig. 9.

The step responses for P control are shown in Fig. 10, where $R = L = 1, 1/T = 0$, and P gains are $k_{1CDM} = 0.4$, $k_{1QDR} = 1$, and $k_{1MSS} = 0.8$ as in Eqs. (27)(29)(34). From Fig. 10, QDR shows about 0.25 decay in one cycle. Compared with CDM, P gain of QDR is 2.5 times, and that of MSS is 2 times higher. Thus QDR and MSS are more oscillatory compared with CDM. This also roughly applies to PI control cases as evidenced in Eqs. (26)(29)(34). Step responses of complementary sensitivity functions are shown in Fig. 11. The QDR and MSS show longer settling time compared with CDM. Thus there is no justification of such gain increase.

## 6. CONCLUSIONS

The important conclusions are summarized as follows.

(1) The time delay can be approximated by a denominator polynomial, as shown below.

$$e^{-Ls} \approx \frac{1}{A_L(s)} = \frac{1}{0.1L^3s^3+0.5L^2s^2+Ls+1}.$$

(2) The discrete sampled-data system is approximated by the continuous system with equivalent time-delay of 0.5~1.5 times of the sampling time $T_s$. Because $T_s \ll L$, the time-delay can be usually neglected.

(3) CDM design of PI control gives satisfactory results with no oscillation in the step response. The PID control is not necessary for such multistage time-lag plants. The P control is usually not recommended, because it lacks the disturbance rejection capability.

(4) Compared with CDM, QDR gains are roughly 2.5 times, and MSS gains are 2 times higher. They show oscillation in step responses as intended in the design purpose of quarter-decay-ratio. Because their settling times are longer, there is no justification of such gain increase.

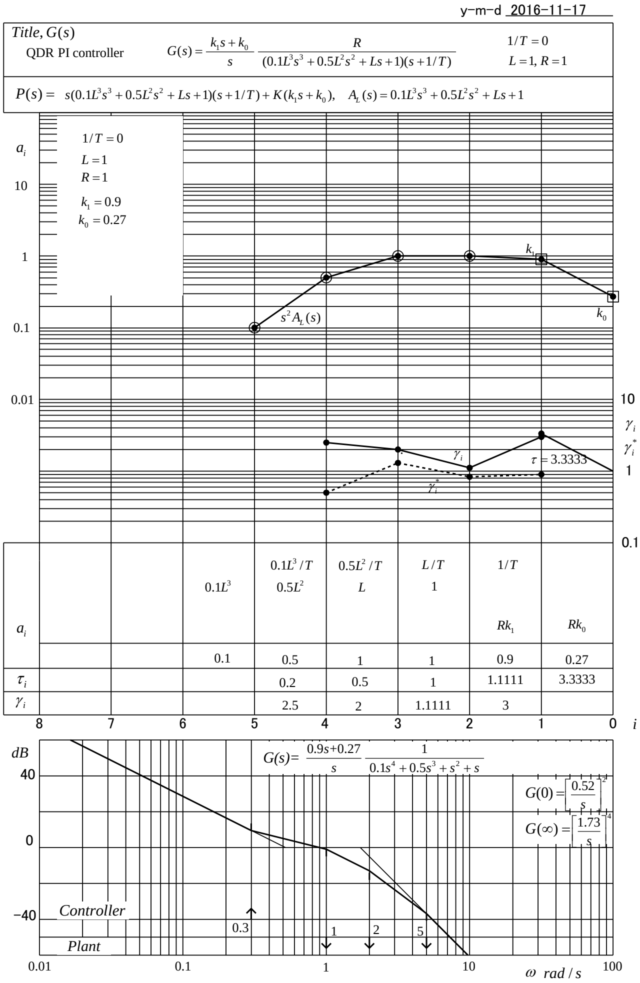


Fig. 6. Coefficient diagram, QDR, $R = L = 1, 1/T = 0$

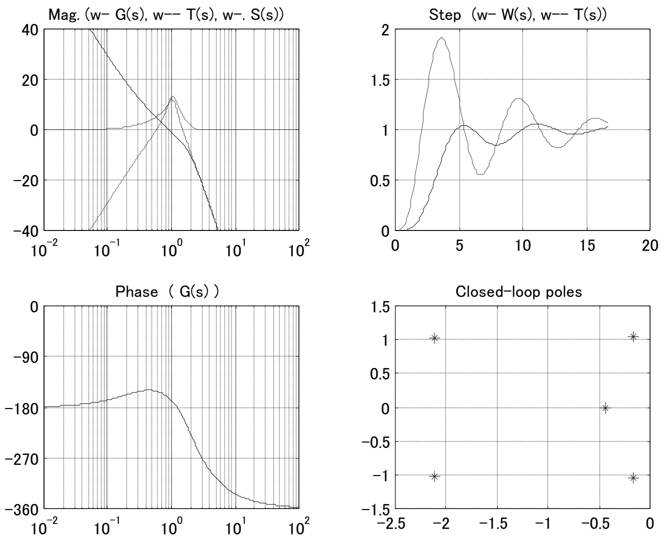


Fig. 7. Responses, QDR, $R = L = 1, 1/T = 0$

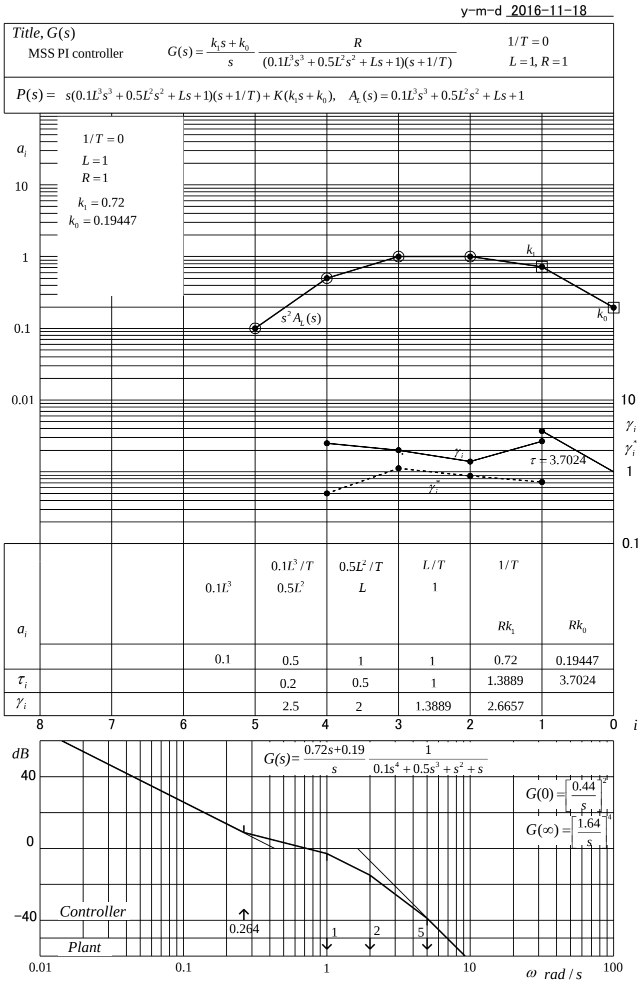


Fig. 8. Coefficient diagram, MSS, $R = L = 1, 1/T = 0$

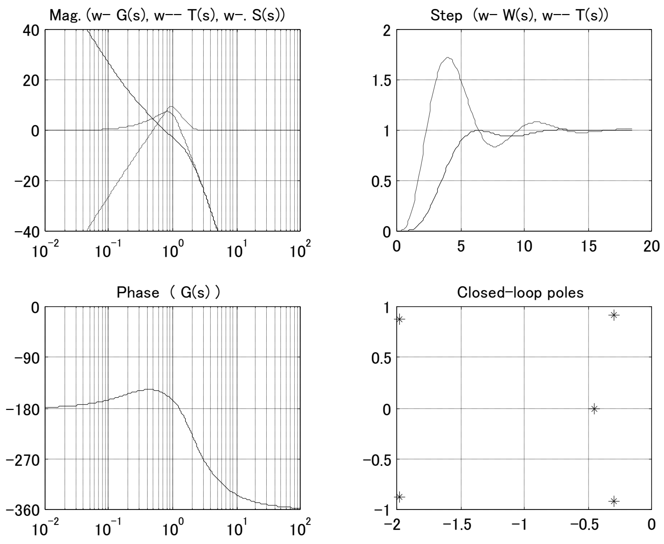


Fig. 9. Responses, MSS, $R = L = 1, 1/T = 0$

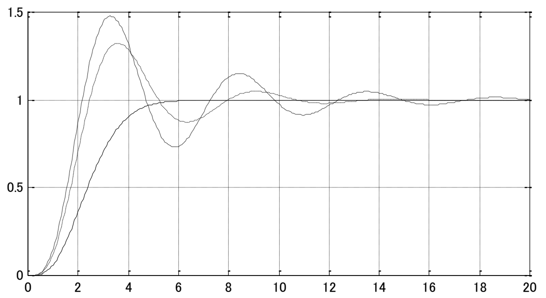


Fig. 10. P control step responses

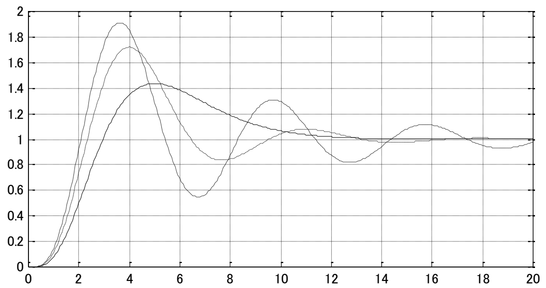


Fig. 10. PI control step response

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